Lecture 10

We begin today's lecture with a simple claim.

Claim. *Let $Q \subseteq \mathbb{R}^n$ be a rectangle and $f, g : Q \to \mathbb{R}$ be bounded functions such that $f \leq g$. Then*

$$\underline{\int_Q} f \leq \underline{\int_Q} g. \tag{3.49}$$

Proof. Let P be a partition of Q, and let R be a rectangle belonging to P. Clearly, $m_R(f) \leq m_R(g)$, so

$$L(f, P) = \sum_R m_R(f) v(R) \tag{3.50}$$

$$L(g, P) = \sum_R m_R(g) v(R) \tag{3.51}$$

$$\implies L(f, P) \leq L(g, P) \leq \underline{\int_Q} g, \tag{3.52}$$

for all partitions P. The lower integral

$$\underline{\int_Q} f \tag{3.53}$$

is the l.u.b. of $L(f, P)$, so

$$\underline{\int_Q} f \leq \underline{\int_Q} g. \tag{3.54}$$

□

Similarly,

$$\overline{\int_Q} f \leq \overline{\int_Q} g. \tag{3.55}$$

It follows that if $f \leq g$, then

$$\int_Q f \leq \int_Q g. \tag{3.56}$$

This is the *monotonicity* property of the R. integral.

3.4 Fubini Theorem

In one-dimensional calculus, when we have a continuous function $f : [a, b] \to \mathbb{R}$, then we can calculate the R. integral

$$\int_a^b f(x)dx = F(b) - F(a), \tag{3.57}$$

where F is the anti-derivative of f.

When we integrate a continuous function $f : Q \to \mathbb{R}$ over a two-dimensional region, say $Q = [a_1, b_1] \times [a_2, b_2]$, we can calculate the R. integral

$$\int_Q f = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x,y)dxdy = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x,y)dxdy \right) \tag{3.58}$$

That is, we can break up Q into components and integrate separately over those components. We make this more precise in the following Fubini Theorem.

First, we define some notation that will be used.

Let $n = k + \ell$ so that $\mathbb{R}^n = \mathbb{R}^l \times \mathbb{R}^\ell$. Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. We can write $c = (a, b)$, where $a = (c_1, \dots, c_)) \in \mathbb{R}^k$ and $b = (c_{k+1}, \dots, c_{k+\ell}) \in \mathbb{R}^\ell$. Similarly, let $Q = I_1 \times \cdots I_n$ be a rectangle in $\mathbb{R}^n$. Then we can write $Q = A \times B$, where $A = I_1 \times \cdots \times I_k \in \mathbb{R}^k$ and $B = I_{k+1} \times \cdots \times I_{k+\ell} \in \mathbb{R}^\ell$. Along the same lines, we can write a partition $P = (P_1, \dots, P_n)$ as $P = (P_A, P_B)$, where $P_A = (P_1, \dots, P_k)$ and $P_B = (P_{k+1}, \dots, P_{k+\ell})$.

Fubini Theorem. *Let $f : Q \to \mathbb{R}$ be a bounded function and $Q = A \times B$ a rectangle as defined above. We write $f = f(x, y)$, where $x \in A$, and $y \in B$. Fixing $x \in A$, we can define a function $f_x : B \to \mathbb{R}$ by $f_x(y) = f(x, y)$. Since this function is bounded, we can define new functions $g, h : A \to \mathbb{R}$ by*

$$g(x) = \underline{\int}_B f_x, \tag{3.59}$$

$$h(x) = \overline{\int}_B f_x. \tag{3.60}$$

Note that $g \le h$. The Fubini Theorem concludes the following: If f is integrable over Q, then g and h are integrable over A and

$$\int_A g = \int_A h = \int_Q f. \tag{3.61}$$

Proof. Let $P = (P_A, P_B)$ be a partition of Q, and let $R = R_A \times R_B$ be a rectangle belonging to P (so R_A belongs to P_A and R_B belongs to P_B). Fix $x_0 \in A$.

First, we claim that

$$m_{R_A \times R_B}(f) \le m_{R_b}(f_{x_0}), \tag{3.62}$$

the proof of which is straightforward.

Next,

$$\begin{aligned} \sum_{R_B} m_{R_A \times R_B}(f) v(R_B) &\le \sum_{R_B} m_{R_B}(f_{x_0}) v(R_B) \\ &= L(f_{x_0}, P_B) \\ &\le \underline{\int}_B f_{x_0} = g(x_0). \end{aligned} \tag{3.63}$$

So,

$$\sum_{R_B} m_{R_A\times R_B}(f)v(R_B) \ \leq g(x_0) \tag{3.64}$$

for all $x_0 \in R_A$. The above equation must hold for the infimum of the r.h.s, so

$$\sum_{R_B} m_{R_A\times R_B}(f)v(R_B) \ \leq m_{R_A}(g). \tag{3.65}$$

Observe that $v(R_A \times R_B) = v(R_A)v(R_B)$, so

$$\begin{aligned} L(f,P) &= \sum_{R_A\times R_B} m_{R_A\times R_B}(f)v(R_A\times R_B) \\ &\leq \sum_{R_A} m_{R_A}(g)v(R_A) \\ &\leq \underline{\int}_A g. \end{aligned} \tag{3.66}$$

We have just shown that for any partition $P = (P_A, P_B)$,

$$L(f,P) \leq L(g,P_A) \leq \underline{\int}_A g, \tag{3.67}$$

so

$$\underline{\int}_Q f \leq \underline{\int}_A g. \tag{3.68}$$

By a similar argument, we can show that

$$\overline{\int}_A h \leq \overline{\int}_Q f. \tag{3.69}$$

Summarizing, we have shown that

$$\underline{\int}_Q f \leq \underline{\int}_A g \leq \overline{\int}_A h \leq \overline{\int}_Q f, \tag{3.70}$$

where we used monotonicity for the middle inequality. Since f is R. integrable,

$$\underline{\int}_Q f = \overline{\int}_Q f, \tag{3.71}$$

so all of the inequalities are in fact equalities. □

Remark. Suppose that for every $x \in A$, that $f_x : B \to \mathbb{R}$ is R. integrable. That's the same as saying $g(x) = h(x)$. Then

$$\begin{aligned}\int_A \left(\int_B f_x \right) &= \int_A dx \left(\int_B f(x,y) dy \right) \\ &= \int_{A \times B} f(x,y) dx dy,\end{aligned} \tag{3.72}$$

using standard notation from calculus.

Remark. In particular, if f is continuous, then f_x is continuous. Hence, the above remark holds for all continuous functions.

3.5 Properties of Riemann Integrals

We now prove some standard calculus results.

Theorem 3.13. *Let $Q \subseteq \mathbb{R}^n$ be a rectangle, and let $f, g : Q \to \mathbb{R}$ be R. integrable functions. Then, for all $a, b \in \mathbb{R}$, the function $af + bg$ is R. integrable and*

$$\int_Q af + bg = a \int_Q f + b \int_Q g. \tag{3.73}$$

Proof. Let's first assume that $a, b \leq 0$. Let P be a partition of Q and R a rectangle belonging to P. Then

$$a m_R(f) + b m_R(g) \leq m_R(af + bg), \tag{3.74}$$

so

$$\begin{aligned} aL(f,P) + bL(g,P) &\leq L(af + bg, P) \\ &\leq \underline{\int}_Q af + bg. \end{aligned} \tag{3.75}$$

Claim. *For any pair of partitions P' and P'',*

$$aL(f,P') + bL(g,P'') \leq \underline{\int}_Q af + bg. \tag{3.76}$$

To see that the claim is true, take P to be a refinement of P' and P'', and apply Equation 3.75. Thus,

$$a \underline{\int}_Q f + b \underline{\int}_Q g \leq \underline{\int}_Q af + bg. \tag{3.77}$$

Similarly, we can show that

$$\overline{\int}_Q af + bg \leq a \overline{\int}_Q f + b \overline{\int}_Q g. \tag{3.78}$$

Since f and g are R. integrable, we know that

$$\overline{\int_Q} f = \underline{\int_Q} f, \ \overline{\int_Q} g = \underline{\int_Q} g. \tag{3.79}$$

These equalities show that the previous inequalities were in fact equalities, so

$$\int_Q af + bg = a \int_Q f + b \int_Q g. \tag{3.80}$$

However, remember that we assumed that $a, b \geq 0$. To deal with the case of arbitrary a, b, it suffices to check what happens when we change the sign of a or b.

Claim.

$$\int_Q -f = -\int_Q f. \tag{3.81}$$

Proof Hint. Let P be any partition of Q. Then $L(f, P) = -U(-f, P)$. □

You should check this claim, and then use it to complete the proof. □