18.100C
Fall 2012

# Practice final exam

No notes, textbooks, calculators, or other material may be used. Please switch off all mobile phones and other electronic devices.

Unless the problem specifically states otherwise, the rules are as follows: you can use any theorem proved in the 18.100C lectures, and any theorem which is in the body of the textbook. However, you should state the theorem clearly when you use it. You may not use any theorems proved in the homework, or ones which are in the problem part of the textbook (if you want to, you have to reproduce their proofs).

1. Take $\mathbb{R}^2$ with the French railroad metric

$$d_{SNCF}(x,y) = \begin{cases} \|x-y\| & x \text{ and } y \text{ lie on the same line through the origin,} \\ \|x\|+\|y\| & \text{otherwise.} \end{cases}$$

   Is this metric space complete? With proof or counterexample, of course.

2. Construct a sequence of real numbers $x_0, x_1, \ldots$ whose set of subsequential limits is precisely $[0,1]$. With proof, of course.

3. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Prove that the function $g(x) = \max\{f(x,y) : y \in [0,1]\}$ makes sense (maximum exists) and is continuous.

4. Let $f$ be a function which is twice differentiable at some point $x_0$. Define another function by

$$g(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & x \neq x_0, \\ f'(x_0) & x = x_0. \end{cases}$$

   Prove that $g$ is differentiable at $x_0$, with derivative $g'(x_0) = f''(x_0)/2$.

5. Take the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x = 1/n, \; n \text{ a natural number,} \\ 0 & \text{for all other } x. \end{cases}$$

   Is it Riemann-integrable? With proofs of course.

6. Let $f : [0,1] \to \mathbb{R}$ be a Riemann-integrable function. Prove that as $n \to \infty$,

$$\lim_{n\to\infty} \int_0^1 f(x)x^n \, dx = 0.$$

Score: 5+5+5+5+5+5 = 30 points.