18.100B Problem Set 5 Solutions

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1) We have $X \subseteq \mathcal{M}$, with $\mathcal{M}$ complete. X is complete if and only if every Cauchy sequence of X converges to some $x \in X$. Let (x_i) be Cauchy, with $x_i \in X$. $\mathcal{M}$ being complete implies that $x_i \to y \in \mathcal{M}$. Therefore y is a limit point of X. So if X is closed, $y \in X$, so every Cauchy sequence converges in X, so X is complete.

   Conversely, suppose X is complete. Therefore every Cauchy sequence of X converges to a point in X. If $y \in X'$, then we can choose a sequence $(x_i) \subseteq X$ with $d(x_i, y) < \frac{1}{i}$, and since this converges to y in $\mathcal{M}$, it is Cauchy in X. Thus by completeness it converges in X, and by the uniqueness of limits, $y \in X$. Therefore $X' \subseteq X$, so X is closed.

2) First, we know that if a sequence converges to some limit L, every subsequence of that sequence converges to L. This implies the "only if" ($\Rightarrow$) of both (a) and (b).

   a) To prove the "if" ($\Leftarrow$) of a), assume the sequences (x_{2n}) and (x_{2n-1}) both converge to the limit L. Then given $\varepsilon > 0$ we can find natural numbers N and N' such that for
   $$n > N, n' > N' \implies |x_{2n} - L| < \varepsilon \text{ and } |x_{2n'-1} - L| < \varepsilon.$$
   Let $M = \max\{2N, 2N'\}$ and notice that if $m > M$, then $|x_m - L| < \varepsilon$ regardless of whether m is even or odd. Therefore (x_m) converges to L.

   b) Here, we reduce to the case of (a). Suppose $x_{2n} \to A$, $x_{2n-1} \to B$, and $x_{5n} \to C$. Consider the sequence (x_{10n}). This is a subsequence of (x_{2n}), so it must converge to A. But is also a subsequence of (x_{5n}), so it must converge to C. By the uniqueness of limits, we have $A = C$. Similarly (x_{10n-5}) is a subsequence of both (x_{2n-1}) and (x_{5n}), so it must converge to both B and C, so $B = C$. Thus $A = B = C$, in particular $A = B$, so now we can apply (a).

3) For any $N \in \mathbb{N}$, $\{x_n + y_n | n > N\} \subseteq \{x_m + y_n | m, n > N\}$. Therefore
   $$\sup_{n>N}(x_n + y_n) \leq \sup_{m,n>N}(x_m + y_n) = \sup_{m>N}(x_m) + \sup_{n>N}(y_n).$$
   Since this is true for all $N \in \mathbb{N}$, it is true in the limit. So $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$. By the above, and since $\liminf(z_n) = -\limsup(-z_n)$ for any bounded sequence z_n,
   $$\begin{aligned}\liminf(x_n + y_n) &= -\limsup(-x_n - y_n)\\ &\geq -\limsup(-x_n) - \limsup(-y_n) = \liminf(x_n) + \liminf(y_n).\end{aligned}$$
   Now we assume (x_n) converges to some L. If $\limsup y_n = \alpha$, then some subsequence $y_{n_k} \to \alpha$. Since $x_n \to L$, any subsequence converges to this limit, so in particular $x_{n_k} \to L$. Therefore the sequence $(x_n + y_n)$ has a subsequence $(x_{n_k} + y_{n_k})$ that converges to $L + \alpha$. Therefore
   $$\limsup(x_n + y_n) \geq L + \alpha = \limsup x_n + \limsup y_n \geq \limsup(x_n + y_n),$$
   so we have equality. Once again the relation between limsup and liminf exploited above shows that equality also occurs for liminf, when (x_n) converges.

4) We have (x_n) a bounded sequence and (a_n) the sequence defined as
   $$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

Let $\alpha = \limsup x_n$, and fix $1 > \varepsilon > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_n < \alpha + \frac{\varepsilon}{2}$, by the definition of $\limsup x_n$. Let B be an upper bound for x_n. Then if $n > \frac{2BN}{\varepsilon}$,

$$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n} = \sum_{i=1}^{N} \frac{x_i}{n} + \sum_{j=N+1}^{n} \frac{x_j}{n} \leq \sum_{i=1}^{N} \frac{B}{\frac{2BN}{\varepsilon}} + \sum_{j=N+1}^{n} \frac{\alpha + \frac{\varepsilon}{2}}{n} \leq \frac{\varepsilon}{2} + \alpha + \frac{\varepsilon}{2} = \alpha + \varepsilon.$$

So for n large enough, $a_n \leq \alpha + \varepsilon$. Therefore $\limsup a_n \leq \alpha + \varepsilon$. But ε can be chosen arbitrarily small, so $\limsup a_n \leq \alpha$. This also shows that $-\limsup -x_n \leq -\limsup -a_n$, or $\liminf x_n \leq \liminf a_n$. So we have

$$\liminf x_n \leq \liminf a_n \leq \limsup a_n \leq \limsup x_n.$$

If $x_n \to x$, then $\liminf x_n = \limsup x_n = x$, so by the inequality $\liminf a_n = \limsup a_n = x$, so $a_n \to x$. However, (a_n) can converge without (x_n) converging. For example, let $x_n = (-1)^n$. Then

$$a_n = \begin{cases} -\frac{1}{n} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases}$$

Since both even and odd subsequences converge to 0, $a_n \to 0$. (x_n), on the other hand, has its odd subsequence converging to -1 and its even subsequence converging to 1 (they are both constant subsequences). So (x_n) does not converge.

5) We have that $0 < x < 1$ and $x_n = 1 - \sqrt{1 - x_{n-1}}$. Since the functions $1 - x$ and $\sqrt{x}$ both take the open interval $(0, 1)$ to itself, by induction $0 < x_n < 1$ $\forall n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

$$1 - \sqrt{1 - x_{n-1}} = x_n \Longrightarrow \sqrt{1 - x_{n-1}} = 1 - x_n \Longrightarrow 1 - x_{n-1} = (1 - x_n)^2 < 1 - x_n$$

since $0 < (1 - x_n) < 1$. This shows that $x_{n-1} > x_n$, so the sequence is decreasing. Therefore the sequence is decreasing and bounded below by 0, so it must have a limit $L \geq 0$. Suppose $L > 0$. Then $(1 - L)^2 < 1 - L < 1$, since it is clear by $L < x_1 < 1$ that $L < 1$ ((x_n) is strictly decreasing, so $L < x_n \forall n \in \mathbb{N}$). Therefore $1 - (1 - L)^2 > L$, so $\exists n \in \mathbb{N}$ such that

$$1 - (1 - L)^2 > x_n \Longrightarrow (1 - L)^2 < 1 - x_n \Longrightarrow 1 - L < \sqrt{1 - x_n} \Longrightarrow L > 1 - \sqrt{1 - x_n} = x_{n+1} > L.$$

This gives $L > L$, a contradiction. Therefore $L = 0$.
Now to calculate the limit of $\frac{x_{n+1}}{x_n}$. We have

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{1 - \sqrt{1 - x_n}}{x_n} = \lim_{n\to\infty} \frac{(1 - \sqrt{1 - x_n}) \cdot (1 + \sqrt{1 - x_n})}{x_n(1 + \sqrt{1 - x_n})} = \lim_{n\to\infty} \frac{1 - (1 - x_n)}{x_n(1 + \sqrt{1 - x_n})}$$
$$= \lim_{n\to\infty} \frac{x_n}{x_n(1 + \sqrt{1 - x_n})} = \lim_{n\to\infty} \frac{1}{1 + \sqrt{1 - x_n}} = \frac{1}{1 + \sqrt{1 - 0}} = \frac{1}{2}.$$

We can substitute the limit of x_n into the limit because the function $\frac{1}{1 + \sqrt{1 - x}}$ is continuous at 0, the limit of (x_n).

6) a) So the defining equation for Φ is

$$\Phi = \frac{a}{b} = \frac{b}{c}$$

where $a = b + c$. So $c \cdot (b + c) = b \cdot b$ which gives

$$\left(\frac{b}{c}\right)^2 = \left(\frac{b}{c}\right) + 1 \Longrightarrow \Phi^2 = \Phi + 1$$

Using the quadratic formula with $x^2 - x - 1 = 0$, which Φ satisfies, we get

$$\Phi = \frac{1 \pm \sqrt{5}}{2}$$

and since Φ was defined to be greater than 1, the $\pm$ sign must be a $+$ sign.

b) We want to show that

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}},$$

where the right hand side is the limit of the sequence x_n where $x_1 = 1$ and $x_n = 1 + \frac{1}{x_{n-1}}$. First we must show that (x_n) converges. It is clear that $x_n > 0$, $\forall n \in \mathbb{N}$. Now this gives that, since $x_n = 1 + \frac{1}{x_{n-1}}$, $x_n \geq 1$, $\forall n \in \mathbb{N}$. And this implies $\frac{1}{x_n} \leq 1$, which gives that $x_{n+1} = 1 + \frac{1}{x_n} \leq 2$. So $1 \leq x_n \leq 2$. Now consider x_{2n-1}, the odd terms of the sequence. The first few are:

$$x_1 = 1 \qquad x_3 = 1 + \frac{1}{1 + \frac{1}{1}} \qquad x_5 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}}$$

Analyzing these, we see that $x_1 < x_3 < x_5 < \ldots$. This is because to get from x_{2n-1} to x_{2n+1}, you add a positive number to an even-numbered denominator of the continued fraction, which makes it greater. Since $x_{2n} = 1 + \frac{1}{x_{2n-1}}$, the even subsequence is 1 more than the inverses of the odd sequence, which is increasing, so the even subsequence is decreasing. Since these subsequences are both bounded between 1 and 2 and are monotonic, they must have limits, say $x_{2n-1} \to a$ and $x_{2n} \to b$. We have

$$a = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} 1 + \frac{1}{1 + \frac{1}{x_{2n-1}}} = 1 + \frac{1}{1 + \frac{1}{\lim x_{2n-1}}} = 1 + \frac{1}{1 + \frac{1}{a}}.$$

$$a(1 + \frac{1}{a}) = (1 + \frac{1}{a}) + 1 \Longrightarrow a + 1 = 2 + \frac{1}{a} \Longrightarrow a^2 = a + 1.$$

But this is exactly the quadratic equation that Φ satisfies. Since $a \geq 1$, $a = \Phi$. Exactly the same argument works for b, since the relation between x_{2n+2} and x_{2n} is identical. Therefore $b = \Phi$, which implies (by problem 2) that $x_n \to \Phi$.

c) Now we want to show that $y_n \to \Phi$ for the sequence (y_n) defined by

$$y_1 = 1, \text{ and } y_n = \sqrt{1 + y_{n-1}}.$$

It is clear that $y_n \geq 1$, since by induction $y_n^2 = 1 + y_{n-1} \geq 2$. Also, $y_1 < \Phi$ and

$$y_{n-1} < \Phi \Longrightarrow y_{n-1} + 1 < \Phi + 1 \Longrightarrow y_n = \sqrt{1 + y_{n-1}} < \sqrt{\Phi + 1} = \Phi.$$

So by induction, $y_n < \Phi$. So $1 \leq y_n < \Phi$ gives that $y_n^2 - y_n - 1 < 0$, or $y_n < \sqrt{1 + y_n} = y_{n+1}$. Thus (y_n) is an increasing sequence that is bounded above, so it converges to some limit, c. We have

$$c = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \sqrt{1 + y_{n-1}} = \sqrt{1 + \lim_{n\to\infty} y_{n-1}} = \sqrt{1 + c}.$$

So now $c = \sqrt{1 + c}$ which gives $c^2 = c + 1$, and since $c > 1$, we once again have $c = \Phi$.

d) Now we define $z_1 = z_2 = 1$, and $z_n = z_{n-1} + z_{n-2}$ for $n > 2$. Let's define $x_n = \dfrac{z_{n+1}}{z_n}$. So we have

$$x_1 = \frac{z_2}{z_1} = \frac{1}{1} = 1, \text{ and } x_n = \frac{z_{n+1}}{z_n} = \frac{z_n + z_{n-1}}{z_n} = 1 + \frac{z_{n-1}}{z_n} = 1 + \frac{1}{x_{n-1}}$$

But this is exactly the same as the (x_n) from a)! Exxxxxcellent... it's all falling into place. We have shown $x_n \to \Phi$, so the ratios of consecutive Fibonacci numbers approaches Φ.