

## 18.100A Fall 2012: Assignment 21

The rules are the same as for previous assignments.

**Reading Mon.: 24.1-24.5** The Euclidean and uniform distances (norms) on  $\mathbf{R}^2$ .

Sequences in  $\mathbf{R}^2$ , limits, continuous functions on  $\mathbf{R}^2$ , Sequential Continuity theorem.

**1.** (1) In the app YouDraw, the mode Manhattan-Skyline allows only continuous paths traced out by a point moving in an  $x$ - $y$  coordinate system, so that its motion is always in the horizontal or vertical direction – i.e., any change of direction is made by a right angle turn.

(i) Write down a suitable definition in  $x$ - $y$  coordinates for the MS-norm  $\| \cdot \|$ , which gives the length of any of the shortest admissible paths connecting  $\mathbf{0}$  and  $\mathbf{x}$ .

Draw the  $\delta$ -neighborhood of  $\mathbf{0}$  in this norm:  $\Delta(\mathbf{0}, \delta) = \{\mathbf{x} \in \mathbf{R}^2 : \| \mathbf{x} \| < \delta\}$ .

(ii) Prove that  $\| \cdot \|$  satisfies the triangle inequality.

**2.** (2) Two norms  $| \cdot |_1$  and  $| \cdot |_2$  in  $\mathbf{R}^2$  are called *equivalent* if there are positive constants  $c$  and  $d$  such that  $| \mathbf{x} |_1 \leq c | \mathbf{x} |_2$  and  $| \mathbf{x} |_2 \leq d | \mathbf{x} |_1$ , for all  $\mathbf{x} \in \mathbf{R}^2$ .

a) Prove that in  $\mathbf{R}^2$  the Euclidean norm  $| \cdot |$  and the uniform norm  $\| \cdot \|$  are equivalent.

If two norms are equivalent, they give the same results when used in definitions or proofs, for example in defining the limit of a sequence, or the continuity of a function  $f(\mathbf{x})$  at a point  $\mathbf{x}_0$ . Two exercises illustrate this: 24.2/3 (lim  $x_n$ ) and 24.4/2 (continuity of  $f(\mathbf{x})$ ).

b) Work 24.2/3, to show that a sequence  $\mathbf{x}_n$  is convergent in the Euclidean norm if and only if it is convergent in the uniform norm.

**3.** (2) Work 24.2/2 to see what convergence of a sequence in the plane looks like.

The output should be a drawing of  $\mathcal{D}$ , with the subregions clearly marked, and arrows on them indicating which limit the points in that subregion are tending to as  $n \rightarrow \infty$ . Pay particular attention to the regions on the boundary of  $\mathcal{D}$ .

**4.** (3: 1, 2)

a) Read the proof of the Bolzano-Weierstrass Theorem for  $\mathbf{R}^2$  (Theorem 24.2C), using coordinate-wise convergence, and then answer Question 24.2/4, if possible without consulting the solution.

b) Prove the theorem by the bisection method: assume the given sequence lies within the box  $B(\mathbf{0}, K)$ , divide the box into equal quarters by a horizontal and a vertical line, and tell which quarter to select the first point  $\mathbf{x}_{n_1}$  from.

Then subdivide similarly this selected quarter, and tell how to select the next point  $\mathbf{x}_{n_2}$  (be careful).

Continuing, you may assume there is only one point  $\mathbf{a}$  inside all of the successively chosen nested squares. Indicate why your subsequence  $\mathbf{x}_{n_i}$  actually is a subsequence, and prove it converges to  $\mathbf{a}$ .

**5.** Work 24.5/5, as a good example of how the sequential continuity theorem in  $\mathbf{R}^2$  is used. Give a direct argument using limits; base your argument on the ideas in 24.1-.5, without using compactness; we will save that for the next assignment.

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18.100A Introduction to Analysis  
Fall 2012

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