18.100A Fall 2011: Assignment 12

Directions: You can collaborate, but must write up the solutions independently, and list collaborators. Consulting solutions to problem sets of previous semesters is not allowed. Cite significant theorems being used in your arguments.

Reading for Mon.: 13.1-.3   The Extremal Point Theorem (Boundedness and Maximum Theorems in the book).

   A continuous function on a sequentially compact interval $[a, b]$ is bounded and has (global) maximum and minimum points on the interval.

   Here "global" is used to distinguish the points from the "local" extrema, also known as the "relative maximum and minimum points" in many calculus courses.

Problem 1. (2)   (a) Work 13.2/1a   (b) Work 13.3/3a
   These give a geometric interpretation and application of the theorem.

Problem 2.   (1)   Find a function $y = f(x)$ whose domain is $[-\pi/2, \pi/2]$, and whose range is the entire y-axis, and prove that any function with these properties cannot also be continuous.

Problem 3. (3: 1,2)
   Assume $f(x)$ is defined, continuous, and positive on $I = [0, \infty)$, and $\lim_{x \to \infty} f(x) = 0$. Prove that $f(x)$ has a maximum point on I.

   The point of this problem is that I is not sequentially compact, so the Extremal Point Theorem is not immediately applicable, and you have to get around that somehow. Do it by the following steps. Part (a) is a warm-up, showing how to deal with I in a simpler argument, and it contributes useful information which will help with Part (b).

   (a) Prove that $f(x)$ is bounded above on I. Deduce that $\bar{m} = \sup_I f(x)$ exists, and give a nontrivial (i.e., a non-zero) lower estimate for it.

   (b) Use your lower estimate to divide I into two pieces: a compact interval at the beginning, followed by an infinite "tail" interval on which – if it exists – the maximum point $\bar{x}$ satisfying $f(\bar{x}) = \bar{m}$ cannot lie; then finish the proof of (b).

Problem 4.   (2)   You can deduce the Extremal Point Theorem from the Boundedness Theorem by a different argument than the one given in the textbook as the proof of Theorem 13.3. This alternative argument is indirect, but gives a shorter proof.

   Assume $f(x)$ is continuous on the compact interval I but has no maximum point $\bar{x}$. Derive a contradiction by considering the properties of the following function:

$$\frac{1}{\bar{m} - f(x)}, \qquad \text{where } \bar{m} = \sup_I f(x) .$$

Indicate the place(s) where you are using the Boundedness Theorem in the argument

   The second part of Assignment 12, based on Wednesday's lecture, is on the second page.

Reading for Wed.: 13.5   Uniform continuity.

Problem 5. (2) Prove that $f(x) = \ln x$ is uniformly continuous on $[1, \infty)$, even though the interval is not sequentially compact, as follows.

Sketch the graph of $f(x)$. For a given $\epsilon > 0$, and pair of points $x_1 < x_2$, if properly drawn the graph should suggest that the δ which works when $x_1 = 1$ will also work for that ϵ when x_1 has any larger value.

The central problem in the calculation is to estimate what δ_ϵ should be in order to make $f(x_2) - f(x_1) \ < \epsilon$; to do that you have to relate the two expressions $f(x_2) - f(x_1)$ and $x_2 - x_1$.

Use the laws of exponentials and logs: exponentiate the inequality, and then subtract 1 to get $x_2 - x_1$ into the proceedings; use the previous comment about the graph to simplify the resulting inequality enough to permit an upper estimation of δ_ϵ.

Problem 6. (2.5: 1.5, 1) Suppose $y = f(x)$ is a function of bounded slope defined on an interval I of any type. This means there is a fixed constant K such that the slope λ of the line joining any two distinct points (x', y') and (x'', y'') on the graph of $f(x)$ satisfies $|\lambda| < K$.

(a) Prove that $f(x)$ is uniformly continuous on I.

(b) A plausible converse is: if $f(x)$ is uniformly continuous on an interval I which we now assume to be compact, then $f(x)$ has bounded slope. Prove this is not true by considering the function $y = \sqrt[3]{x}$ on $[-1, 1]$.

Problem 7. (2.5: 1.5, 1) Work P13-6 a,b. This ties together uniform continuity with the earlier notion of Cauchy sequence.

(For part (b), you can use the result in Assignment 5, Problem 3a: if a sequence $\{x_n\}$ converges, then it is a Cauchy sequence.)

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