Lecture 2: Q&A

Gaussian integral

1 *How do you derive* $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$*?*

For that story see Section 7.2 in the notes (page 40ff).

2 *What if* $\alpha = -1$ *in the integral* $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx$*?*

Then the exponent becomes e^{x^2}, so the exponential, and its integral, blows up as $|x| \to \infty$.

3 *For* $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx$*, why can't you choose* $[x] = \mathrm{L}^2$ *with* $[\alpha] = \mathrm{L}^{-4}$*?*

You can! You then get the same conclusion: that the integral is proportional to $\alpha^{-1/2}$. Indeed you can choose any dimensions for x, so long as x is not dimensionless. You can even choose x to be dimensionless; if you do, however, the method of dimensions will not help you guess the integral.

4 *Why use* α *in* $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx$*?*

Without α, the exponent is $-x^2$, and it has to be dimensionless. So x would be dimensionless; the method of dimensions then cannot help you guess the integral.

5 *I understood* $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx \sim \alpha^{-1/2}$ *but got lost at the end.*

The last step is figuring out the constant of proportionality. I didn't derive it in lecture, and instead just used the result (explained in Section 7.2 of the notes) that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Putting in $\alpha = 1$ turns $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx$ into $\int_{-\infty}^{\infty} e^{-x^2}\,dx$, so to get $\sqrt{\pi}$ as the result, the full integral has to be

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}.$$

6 *How can I know right away that* $f(\alpha) \neq \mathrm{L}^3\alpha$ *rather than* $k/\sqrt{\alpha}$*?*

It's fine to say that $f(\alpha) \neq \mathrm{L}^3\alpha$. But how do you make a quantity with dimensions L^3? All you can use in your construction is α. Since $[\alpha] = \mathrm{L}^{-2}$, the L^3 must come from $A\alpha^{-3/2}$, where A is a (dimensionless) constant. Thus $\mathrm{L}^3\alpha$ is $\alpha^{-1/2}$ times a dimensionless constant.

Dimensions

7 *It is very strange that the integral of x, where $[x] = \mathrm{L}$, is also a length.*

As another example, here is the relation between distance and velocity:

$$s = \int v\,dt.$$

The integral is the 'area' under the v curve. The vertical axis, which is velocity, has dimensions of LT^{-1}. The horizontal axis, which is time, has dimensions of T. So the 'area' has dimensions of $\mathrm{LT}^{-1}\mathrm{T}$, which is a length. And it should be a length, since the integral gives a distance.

8 *Why does $\int x^{-\alpha x^2}\,dx$ become impossible to integrate when x has dimensions?*

The same problem happens when doing a sum instead of an integral, and is easier to see the problem in that case. Consider this sum:

$$\sum x^{-\alpha x^2}.$$

The exponent $-\alpha x^2$ is dimensionless. Let's say that, to do the sum, you evaluate the terms when $\alpha x^2 = 1, 2, 3, \ldots$. There is no problem if x is dimensionless. However, now let x have dimensions of length. Then the first term, which is x^{-1}, has dimensions of L^{-1}. The second term, which is x^{-2}, has dimensions of L^{-2}. And so on. These terms, which have to be added together, do not have the same dimensions, which is the problem.

The same problem happens with the integral, which is just a continuous sum.

Pyramid

9 *For the truncated pyramid's volume, why not use $h \to 0$ as another easy case?*

That's a good suggestion, which I incorporated into the derivation on Friday. Thanks!

10 *Would $h \to \infty$ be an easy case for the pyramid?*

That's also a good suggestion, which I incorporated into the derivation on Friday. Thanks!

11 *I'm confused about the tests $a \to 0$, $b \to 0$.*

The test $a \to 0$ is the special case of a normal pyramid (not truncated), with a square base and a point above it. Whatever formula we invent for the truncated pyramid must work for the normal pyramid, which is the special case $a \to 0$ of the truncated pyramid. The test $b \to 0$ is the same case with the pyramid flipped over (so the point points downward).

12 *How was $V = \frac{1}{3}h(a^2 + b^2)$ derived?*

By guessing a formula that would work when $a \to 0$ or when $b \to 0$. For $a \to 0$, a valid formula is $hb^2/3$. For $b \to 0$, a valid formula is $ha^2/3$. The combination $V = \frac{1}{3}h(a^2 + b^2)$ works in both cases.

13 *Where does the one-third come from in the pyramid volume?*

From the very special case $a = 0$, $b = 2$, $h = 1$. You take six of those pyramids and form a cube of side 2 and volume 8. So you know the volume of each pyramid (it is 4/3). The only way to get the volume correct is to use $V = hb^2/3$. (See Section 8.5.)

14 *How do you use the $a = 0$, $b = 0$, $a = b$ cases to find the volume of the truncated pyramid?*

I hope that the summary on Friday or Section 8.3 of the notes clears that up. But ask again if it is not clear.

15 *Why is $V \propto h$?*

The volume should be proportional to the height because (and this is a rough argument) it usually is. Imagine a cylinder of radius r and height h. If you stack two of those cylinders to make a longer one, with height $2h$, then you've doubled the volume. So its volume is proportional to h. And that relation is true for the pyramid as well.

16 *Why, when $a = b$, do you get a cube rather than a sheet?*

It's a cube if $a = h$ as well. In general it is a rectangular prism: It has a square base and square top, each with side b (or a), and it has height h.

General

17 *How do you figure out what equation to start with when figuring out equations from dimensional analysis and easy cases (such as for the pyramid)? What is the best way to start the easy cases method?*

Start with the easiest, most familiar cases. For example, $a = 0$ is good when analyzing the truncated pyramid because it produces a familiar shape (the ordinary pyramid).

18 *How do you guess the constant term when there is no easy case (for example, for $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx$)?*

You usually cannot guess it without an easy case. But there is almost always an easy case! And if the easy case is not easy enough to solve analytically, you can sometimes find its value from experiment. An example is the drag coefficient of the cones. The argument in lecture will show that the drag coefficient is constant for the easy case of high-speed flow. But our mathematical knowledge is not advanced enough, and may never be advanced enough, to deduce the constant from the Navier–Stokes equations. So instead we measure the constant, for example in a wind tunnel.

19 *If the constant is not known, should it just be carried through as a constant such as C? What if there is more than one constant?*

That's a good idea to use C. It's what we do with the drag coefficient c_{d}. In most cases we cannot calculate it, so we just use the symbol, and compute it from experiment.

20 *We are spending too much time on dimensions!*

You are probably right, which is why I went ahead with easy cases without answering the above questions in class.