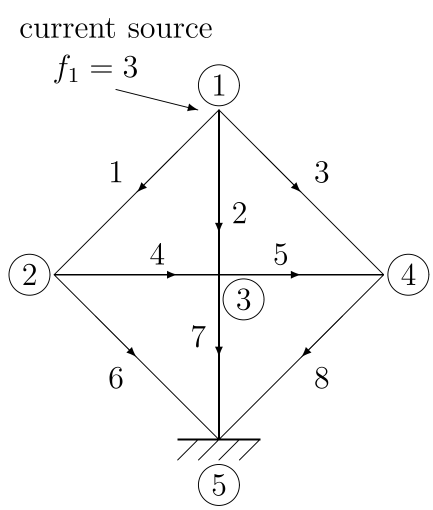
18.085 Computational Science and Engineering I
Fall 2008

18.085 Quiz 2 November 3, 2004 Professor Strang

Your name is: SOLUTIONS

Grading 1.
2.
3.

1) **(36 pts.)** The 5 nodes in the network are at the corners of a *square* and the center. Node 5 is grounded so $x_5 = 0$. All 8 edges have conductances $c = 1$ so $C = I$.



$$A_{8\times 4} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(a) Fill in the 8 by 4 incidence matrix A (node 5 grounded). What is $A^{\mathrm{T}}A$? Is $A^{\mathrm{T}}A$ invertible (**YES** ,NO)?

$$A^{\mathrm{T}}A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}$$

(b) How many independent solutions to $A^{\mathrm{T}}y = 0$? **4**. Write down *one nonzero solution* y.

Ans. The upper left loop gives $y = (1, -1, 0, 1, 0, 0, 0, 0)$

(c) The current source $f_1 = 3$ enters node 1 and exits at grounded node 5. In 2 by 2 *block form* (using A), what are the 12 equations for the 8 currents y and the 4 potentials x?

$$\begin{bmatrix} I & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ f \end{bmatrix} \quad \text{with} \quad b = 0 \quad \text{and} \quad f = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(d) Write out *in full with numbers* the 4 equations for the 4 potentials, after the currents y are eliminated. Using symmetry (or guessing or solving) what is the solution x_1, x_2, x_3, x_4?

Ans. The equations are $A^{\mathrm{T}}Ax = f$ and the solution is $x = (2, 1, 1, 1)$. Unit currents flow to x_5 on edges 1–6 and 2–7 and 3–8. Voltage drop $= 1$ on those six edges.

2) **(24 pts.)** The same 8 edges and 5 nodes form a square pin-jointed truss. The pin at node 5 is held in position so $x_5^{\mathrm{H}} = x_5^{\mathrm{V}} = 0$. All 8 elastic constants are $c = 1$ so $C = I$.

(a) How many unknown displacements? **8**

What is the shape of the matrix A in $e = Ax$? **8 by 8**

Find the *first column* of A, corresponding to the stretching e in the 8 edges from a small displacement x_1^{H} at node 1.

$$\begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ +\sqrt{2}/2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) Are there any nonzero solutions to $Ax = 0$? (**YES** ,NO)

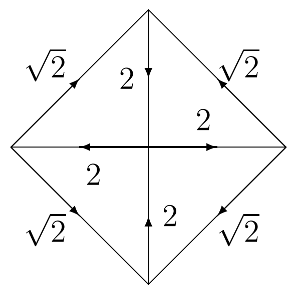
How many independent solutions do you physically expect? **1**

Draw a picture of each independent solution (if any) to show the movement of the 4 nodes.

Ans. Rotation around node 5 has $x = (2, 0, 1, 1, 1, 0, 1, -1)$.

(c) How many independent solutions to $A^{\mathrm{T}}y = 0$? Can you find them?

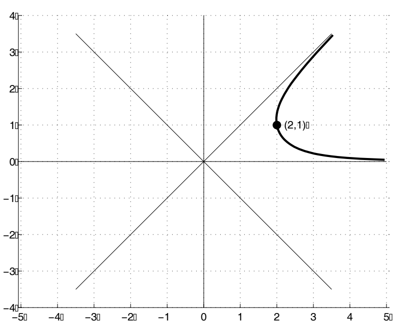
Ans. Since A is square, there will be one line of solutions to $A^{\mathrm{T}}y = 0$ when there is one line of solutions to $Ax = 0$ (Rank 7). The equations $A^{\mathrm{T}}y = 0$ look for a set of bar forces that balance themselves! One set is drawn here:



3) **(40 pts.)** (a) Find a 4th degree polynomial $s(x, y)$ with only 2 terms that solves Laplace's equation. Please draw a box around your answer $s(x, y)$.

Ans. $(x + iy)^4$ gives $s(x, y) = \boxed{4x^3y - 4xy^3}$.

(b) In the xy plane draw all the solutions to $s(x, y) = 0$. Then in the same picture *roughly* draw the curve $s(x, y) = c$ that goes through the particular point $(x, y) = (2, 1)$.



Ans. $4x^3y = 4xy^3$ gives $x = 0$ or $y = 0$ or $x = \pm y$ (four lines). Through $x = 2, y = 1$ will go the curve $s(x, y) = 4 \cdot 8 - 4 \cdot 2 = 24$. It won't cross the lines because they have $s(x, y) = 0$. It will get close to the lines $y = 0$ and $x = y$ as x gets large, because $4x^3y - 4xy^3 = 24$ gives $xy(x + y)(x - y) = 6$. If x and $x + y$ get large then either y or $x - y$ must get small! The curve isn't a hyperbola, I think it must be symmetric across the line $\theta = \pi/8$.

(c) If the curves $s(x, y) = c$ are the *streamlines* of a potential flow (in the usual framework), what is the corresponding velocity $v(x, y) = w(x, y)$?

$$w(x, y) = \left(\frac{\partial s}{\partial y}, -\frac{\partial s}{\partial x}\right) = \left(4x^3 - 12xy^2, 4y^3 - 12yx^2\right) .$$

(d) (this Green's formula question is *not* related to parts a, b, c)

Suppose $w(x, y) = (w_1(x, y), 0)$ is a flow field. With $w_2 = 0$ write down the remaining (not zero) terms in Green's formula for the integral $\iint (\text{grad } u) \cdot w \, dx \, dy$ in the unit square $0 \le x \le 1, 0 \le y \le 1$. Substitute for n and ds when you know what they are for this square.

Ans. Green's formula in the plane is

$$\iint (\text{grad } u) \cdot w \, dx \, dy = -\iint u \, \text{div} \, w \, dx \, dy + \int u \, w \cdot n \, ds \, .$$

Here $w_2 = 0$ and $n = (1, 0)$ on the right side and $n = (-1, 0)$ on the left side. This leaves

$$\iint \frac{\partial u}{\partial x} w_1 \, dx \, dy = -\iint u \frac{\partial w_1}{\partial x} \, dx \, dy + \int_{\text{up right side}} u \, w_1 \, dy - \int_{\text{up left side}} u \, w_1 \, dy$$

(e) A one-dimensional formula on any horizontal line $y = y_0$ is integration by parts:

$$\int_{x=0}^{1} \frac{du}{dx} w_1(x) \, dx = -\int_{x=0}^{1} u(x) \frac{dw_1}{dx} \, dx + uw_1(x = 1) - uw_1(x = 0) \, .$$

Here u and w_1 are $u(x, y_0)$ and $w_1(x, y_0)$ since $y = y_0$ is fixed.

Question 1 How do you derive your Green's formula in part (d) from this one-dimensional formula? ANSWER IN ONE SENTENCE, NO MATH SYMBOLS !!

Ans. Integrate the 1D formula from $y = 0$ to $y = 1$.

Question 2 (not related) Find all vector fields of this form $(w_1(x, y), 0)$ that can be velocity fields $v = w = (w_1(x, y), 0)$ in potential flow [so $v = \text{grad } u$ and $\text{div } w = 0$ as usual].

Ans. Potential flow with $w = (w_1(x, y), 0)$ requires

$$\text{div } w = \frac{\partial w_1}{\partial x} = 0 \quad \text{and also} \quad w_1(x, y) = \frac{\partial u}{\partial x} \, .$$

Then $w_1 = \text{constant}$! The only horizontal potential flow is *uniform flow.*