Fourier Sine Series

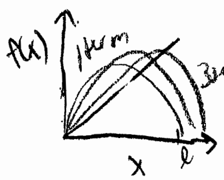
S.L problem: $\begin{cases} \frac{d^2y}{dx^2} + \lambda y = 0 & 0 \le x \le \ell \\ y(0) = 0 = y(\ell) \end{cases}$ $\quad p=1,\ r=0,\ q=1$ "proper"

seen before: $y_n(x) = B_n \underbrace{\sin\left(\frac{n\pi x}{\ell}\right)}_{\varphi_n(x)}$, $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$ $n = 1, 2, 3 \ldots$

$$\int_a^b r(x)\varphi_n(x)\varphi_m(x)\,dx = \int_0^\ell 1 \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)dx = \begin{cases} \frac{\ell}{2} & \text{if } n = m \\ 0 & \text{if } n \ne m \end{cases}$$

$$\left(\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \ne \left(\frac{m\pi}{\ell}\right)^2 = \lambda_m\right)$$

ex $\quad f(x) = x \qquad 0 \le x \le \ell$



$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{\ell}\right) \qquad B_n = \frac{2}{\ell}\int_0^\ell dx\ x\sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{2}{\ell}\left(\left.\frac{-x\cos\left(\frac{n\pi x}{\ell}\right)}{n\pi/\ell}\right|_0^\ell + \int_0^\ell \frac{\cos\left(\frac{n\pi x}{\ell}\right)}{n\pi/\ell} = \frac{2}{\ell}\cos(n\pi + 0)\right)$$

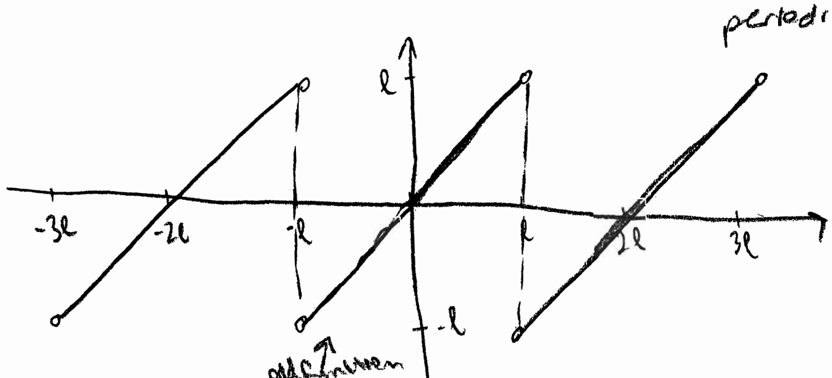
$$B_n = \frac{2\ell}{n\pi}(-1)^{n-1}$$

$$\Rightarrow x = \sum_{n=1}^{\infty} \frac{2\ell}{n\pi}(-1)^{n-1}\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\hat{f}(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{\ell}\right), \quad B_n = \frac{2\ell}{2\pi}(-1)^{n-1}$$

What does this series converge to?
When we extend x to real line $-\infty < x < \infty$
- $\hat{f}(x)$ is periodic (period 2ℓ)
- is odd.

$\hat{f}(x)$ is called the "odd periodic extension" of $f(x)$



More generally,

if $f(x)$ is continuous at x, then $f(x) = \hat{f}(x)$

if $f(x)$ is discontinuous at $x = x_0$, then $f(x) = \frac{1}{2}[f(x_0^+) + f(x_0^-)]$

$f(x_0^{\pm}) = \lim_{x \to x_0^{\pm}} f(x)$

Fourier Cosine Series

motivation: SL problem $y''(x) + \lambda y = 0 \quad 0 \le x \le \ell$

$y'(0) = 0 = y'(\ell)$ (homogeneous B.C.)

solutions: $y(x) = y_n(x) = A_n \cos\left(\frac{n\pi x}{\ell}\right) \; n = 0, 1, 2, 3 \quad \lambda = \lambda_n = \left(\frac{n\pi}{\ell}\right)^2$

Given any "admissible" $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right), \quad A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right)$$

to find A_0: $\int_0^{\ell} f(x)\,dx = A_0 \ell + \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\Big|_0^{\ell} \longrightarrow A_0 = \frac{1}{\ell}\int_0^{\ell} f(x)\,dx$ (average of $f(x)$)

$$\int_0^{\ell} f(x) \cos\left(\frac{m\pi x}{\ell}\right) dx = \sum_{n=0}^{\infty} A_n \int_0^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right) dx$$

use $\int_0^{\ell} \cos\left(\frac{m\pi x}{\ell}\right) \cos\left(\frac{n\pi x}{\ell}\right) dx = \begin{cases} \ell/2 & m = n \\ 0 & m \neq n \end{cases}$

$$A_m = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\frac{m\pi x}{\ell}\right) dx, \quad m \neq 0$$