Bessel equation: $x^2 Z_p''(x) + x Z_p'(x) + (x^2 - p^2) Z_p(x) = 0$

define $w(x) = Z_p(ix)$ $\quad (i^2 = -1)$

ODE for w: $x^2 w'' + x w' - (x^2 + p^2) w = 0$

Modified Bessel equation

$$w(x) \begin{cases} c_1 J_p(ix) + c_2 J_{-p}(ix) & p \neq \text{integer} \\ c_1 J_n(ix) + c_2 Y_n(ix) & p = n = \text{integer} \end{cases}$$

$$J_p(\underset{\xi}{ix}) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\overset{ix}{\xi}/2\right)^{2k+p}}{k!\,\Gamma(k+p+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k\, \overset{(-1)^k}{i^{2k+p}}\, (x/2)^{2k+p}}{k!\,\Gamma(k+p+1)} = i^p \sum_{k=0}^{\infty} \frac{(x/2)^{2k+p}}{k!\,\Gamma(k+p+1)}$$

If x: real, this is real

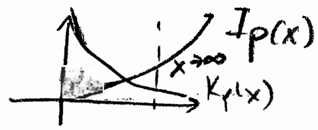
Define $I_p(x) = i^{-p} J_p(ix) = \underbrace{\sum_{k=0}^{\infty} \frac{(x/2)^{2k+p}}{k!\,\Gamma(k+p+1)}}_{\text{real if } x \text{: real}}$ → Modified Bessel function of 1st kind

$K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix)$ → Modified Bessel function of 2nd kind

$\boxed{W(x) = c_1 I_p(x) + c_2 K_p(x)}$: general solution of modified Bessel equation

$x \to \infty$

$$I_p(x) \simeq \frac{e^x}{\sqrt{2\pi x}}$$



$$K_p(x) \simeq \sqrt{\frac{\pi}{2x}}\, e^{-x}$$

$x \to 0$

$$I_p(x) \simeq \frac{1}{2^p p!} x^p$$
$$I_{-p}(x) \simeq \frac{1}{2^p \Gamma(1-p)} x^{-p}$$
$$K_p(x) \simeq 2^{p-1} \Gamma(p)\, x^{-p} \quad (p \geq 1), \quad K_0(x) \simeq -\ln x$$

Special case: p: half-integer $= n + \frac{1}{2}$, n: integer
→ Bessel functions become elementary

ex $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, $\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$
$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$ $\quad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x$

Generally, $J_{n+\frac{1}{2}}(x) = \frac{2n-1}{x} J_{n-\frac{1}{2}}(x) - J_{n-\frac{3}{2}}(x)$

$$I_{n+\frac{1}{2}}(x) = -\frac{2n-1}{x} I_{n-\frac{1}{2}}(x) + I_{n-\frac{3}{2}}(x)$$