ex $\quad x^2y'' + (x^2+x)y' - y = 0 \qquad x_0 = 0$: regular singular point

$$y'' + \frac{x^2+x}{x^2}y' - \frac{y}{x^2} = 0$$

$$y'' + \frac{1}{x}\frac{x^2+x}{x}y' - \frac{1}{x^2}y = 0$$

$R=1 \quad P = x+1 \quad Q \equiv 1$

Case $s_1 = s_2$: Frobenius method gives 1 solution

ex (cont.) $\quad y = x^s \sum_{k=0}^{\infty} A_k x^k = \sum_{k=0}^{\infty} A_k x^{k+s}$; find A_k and s

Recall: We found 1 solution (only) of form $\sum_{k=0}^{\infty} a_k x^k$

$$y' = \sum_{k=0}^{\infty}(k+s)A_k x^{k+s-1}, \quad y'' = \sum_{k=0}^{\infty}(k+s)(k+s-1)A_k x^{k+s-2}$$

$$(x^2+x)y' = \sum_{k=0}^{\infty}(k+s)A_k x^{k+s+1} + \sum_{k=0}^{\infty}(k+s)A_k x^{k+s}$$

$$x^2y'' = \sum_{k=0}^{\infty}(k+s)(k+s-1)A_k x^{k+s}$$

$$\sum_{k=0}^{\infty}(k+s)(k+s-1)A_k x^{k+s} + \sum_{\substack{k=0\\k=1}}^{\infty}(k-1+s)A_{k-1}x^{k+s} + \sum_{k=0}^{\infty}(k+s)A_k x^{k+s} - \sum_{k=0}^{\infty}A_k x^{k+s} = 0$$

$$\underbrace{\left[s(s-1)A_0 + sA_0 - A_0\right]}_{\text{take } k=0 \text{ (lowest power of x)}}\cancel{x^s} + \sum_{k=1}^{\infty}\left\{(k+s)(k+s-1)A_k + (k-1+s)A_{k-1} + (k+s)A_k - A_k\right\}\cancel{x^{k+s}} = 0$$

$$\begin{cases} s(s-1)\cancel{A_0} + (s-1)\cancel{A_0} = 0 \rightarrow (s-1)(s+1) = 0 \rightarrow \boxed{s = \pm 1} \qquad A_0 \neq 0 \\ (k+s)(k+s-1)A_k + (k-1+s)A_{k-1} + \underbrace{(k+s)A_k - A_k}_{k+s-1} = 0, \quad k \geq 1 \end{cases}$$

$\left.\begin{matrix} s_1 = 1 \\ s_2 = -1 \end{matrix}\right\} \; s_1 - s_2 = 2$: integer (>0)

$$(k+s-1)[(k+s)A_k + A_{k-1} + A_k] = 0 \qquad k \geq 1$$

$\underline{s = s_1 = 1:}$ $\quad \cancel{k} \cdot [(k+2)A_k + A_{k-1}] = 0 \rightarrow \boxed{A_k = \frac{A_{k-1}}{k+2}}$

$k=1: \quad A_1 = -A_0/3$

$k=2: \quad A_2 : A_0/3\cdot 4 \qquad y(x) = 2A_0 \dfrac{e^{-x} - 1 + x}{x}$ found it already by $\sum_{n=0}^{\infty} a_n x^n$

Frobenius series is Taylor series

$\underline{S = S_2 = -1:}$ $\quad (k-2)(kA_k + A_{k-1}) = 0 \qquad k \geq 1 \qquad s_1 - s_2 = 2$

$k=1: \quad A_1 + A_0 \Rightarrow A_1 = -A_0 \rightarrow A_0$: arbitrary

$k=2: \quad 0\cdot(2A_2 + A_1) = 0 \rightarrow A_2$ is arbitrary

$k \geq 3: \quad A_k = \dfrac{-A_{k-1}}{k}$ if it succeeds, use s_2.

$$y(x) = x^s \sum_{k=0}^{\infty} a_k x^k$$

$$= \frac{1}{x}\Big\{ A_0 \underbrace{-A_0}_{A_1} x + \underbrace{\boxed{A_2 x^2 + A_3 x^3 + \ldots}}_{\substack{\text{depends only on } A_2 \\ (\text{proportional to } A_2)}} \Big\}$$

$$y(x) = \frac{A_0(1-x)}{x} + \frac{1}{x}\{A_2 x^2 + A_3 x^3 + \ldots\}$$

$$= A_0 u_2(x) + A_2 \underbrace{u_1(x)}_{\text{independent of } A_2}$$

$$u_2(x) = \frac{1-x}{x}$$

Set $A_2 = 0$: $\quad y(x) = C_2 \dfrac{1-x}{x} \qquad$ c is arbitrary

general solution: $\quad y(x) = C_1 \dfrac{e^{-x} - 1 + x}{x} + C_2 \dfrac{1-x}{x}$