2nd-order ODE
linear, homogeneous

$$y''(x) + A_1(x)y'(x) + A_2(x)\,y(x) = 0$$

$\uparrow$ coefficient is $\underline{\underline{1}}$!

<u>Classification of point x_0:</u>

(i) x_0: (regular) ordinary point if $A_1(z)$, $A_2(z)$: analytic at $z = x_0$

• in this case, $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \rightarrow$ get 2 independent solutions

(ii) x_0: regular singular point if A_1 or A_2 not analytic, but $(z-x_0)A_1(z)$, $(z-x_0)^2 A_2(z)$: analytic near x_0

(iii) x_0: irregular singular point if x_0 is not ordinary or regular singular.

<u>Frobenius Method</u>: well-suited for points (ii)

"canonical form" of ODE: (for $x_0 = 0$)

$$R(x)y'' + \frac{1}{x}P(x)y' + \frac{1}{x^2}Q(x)y = 0$$

R, P, Q: analytic at $x_0 = 0$

Without loss of generality, take $x_0 \equiv 0$.

$\boxed{R(0) \neq 0}$ all x in $(\cancel{x_0} - \delta, \cancel{x_0} + \delta)$

$R_0 = 1$ (make it 1)

$$A_1(z) = \frac{1}{x}\frac{P(z)}{R(z)}, \quad A_2(z) = \frac{1}{x^2}\frac{Q(z)}{R(z)}$$

$x_0 = 0$ can be ordinary or regular singular point of ODE

Frobenius Method: $y(x) = x^s \sum_{n=0}^{\infty} a_n x^n$ find s, a_n ($a_0 \neq 0$)

s can be complex

$$\overset{\text{ODE}}{\Longrightarrow} f(s)a_0 x^{s-2} + [f(s+1)a_1 + g_1(s+1)a_0]x^{s-1} + \dots + \left[f(s+k)a_k + \sum_{n=1}^{k} g_n(s+k)a_{k-n}\right]x^{s+k-2} + \dots = 0$$

$$f(s) = s(s-1) + P_0 s + Q_0 \qquad g_n(s) = R_n(s-n)(s-n-1) + P_n(s-n) + Q_n$$

x^{s-2} term: $f(s) = 0 \longrightarrow s_{1,2} = \frac{1-P_0}{2} \pm \frac{1}{2}\sqrt{(1-P_0)^2 - 4Q_0}$ 2 roots

indicial eqn $\qquad s_1, s_2$

<u>Theorem</u>: • if $s_1 \neq s_2$, $s_1 - s_2 \neq$ integer $\rightarrow$ 2 independent solutions

• if $s_1 \neq s_2$, $s_1 - s_2 =$ integer $> 0 \rightarrow$ 1 or 2 solutions of Frobenius form

• if $s_1 = s_2 \rightarrow$ 1 solution of Frobenius form

(i) $s_1 \neq s_2$, $s_1 - s_2 \neq$ integer $s = s_1$ or s_2, $s_1 - s_2 \neq k$

$\underline{x^{s-1} \text{ term}}$: $f(s+1)a_1 + g_1(s+1)a_0 = 0 \rightarrow a_1 = \cdots$ if $f(s+1) \neq 0$

$\vdots$

$\underline{x^{s+k-2} \text{ term}}$: $f(s+k)a_k + \sum_{n=1}^{k} g_n(s+k)a_{k+n} = 0 \rightarrow a_k = \cdots$ if $f(s+k) \neq 0$

$a_{k-1} \cdots a_0$

can never be 0

$\rightarrow$ 2 independent solutions $\begin{cases} s = s_1 \rightarrow y(x) = a_0 u_1(x) \\ s = s_2 \rightarrow y(x) = a_0 u_2(x) \end{cases}$

(ii), (iii): exceptional cases (ii) $s_1 - s_2 =$ Integer > 0, (iii) $s_1 = s_2$

$f(s) = (s - s_1)(s - s_2)$ $f(s_1 + k) = k[k + (s_1 - s_2)]$

$f(s_2 + k) = k[k - (s_1 - s_2)]$

- s_1: imaginary $\rightarrow$ s_2: imaginary $\rightarrow$ $s_1 - s_2$: imaginary $\therefore f(s+k) \neq 0$

$s_2 = \bar{s}_1$

- s_1, s_2: real, $s_1 > s_2$ $f(s_1 + k) \neq 0$

$s_1 - s_2 > 0$ $f(s_2 + k) = 0$ when $s_1 - s_2 = k$

- $\underline{\text{Case } s_1 - s_2 = m > 0,\ s_1, s_2: \text{real},\ s_1 > s_2}$ (1 or 2 solutions) 1 solution for $\underline{s_1}$, or both

$k = m$: $\underbrace{f(s_2 + m)}_{0} a_m + \sum_{n=1}^{m} g_n(s_2 + m)a_{m-n} = 0$

$s = s_2$:

I get 2 solutions when $\sum_{n=1}^{m} g_n(s + m)a_{m-n} = 0$

2 arbitrary constants: a_0, a_m (general solution)