Elementary Functions of One Complex Variable

$$z \xrightarrow[\text{mapping}]{f} w$$

variable $\quad$ $f(z)$

1. integral-power function: $f_{(n)}\, z = z^n, \quad n = 0, 1, 2, \ldots$

   definition (recursively): $\begin{cases} f_{(0)}(z) = 1 \\ f_{(n+1)}(z) = f_{(n)}(z) \cdot z \end{cases}$

   $$z^n = (x+iy)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos\theta + i\sin\theta)^n$$

2. polynomial function:

   $$P_m(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0 = \sum_{n=0}^{m} a_n z^n$$

   degree

   complex coefficients

   alternatively, $P_m(z) = \sum_{n=0}^{m} a_n (z - z_0)^n$

   constant complex number

3. rational functions: ratios of polynomials

   $$\frac{P_n(z)}{Q_m(z)} \qquad z: Q_m(z) \neq 0$$

4. power series: (2) with $m \to \infty$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

- if $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, finite or 0, then the series converges only for $|z - z_0| < \frac{1}{L}$.

5. exponential function:

Suppose z is real. $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$

$n! = 1 \cdot 2 \cdot 3 \cdots n$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}} \to \text{converges for all } x$$

$$e^{x_1 + x_2} = e^{x_1} e^{x_2} \qquad (e^x)^\alpha = e^{\alpha x} \qquad \alpha, x, x_1, x_2 : \text{real}$$

Generalize to complex variable:

$$S(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!} + \dots = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!}$$

Does this converge for all z?

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 = L \qquad z - \overset{0}{z_0} < \frac{1}{L} = \infty$$

$\therefore$ the series converges for all z

$$e^z = S(z)$$

6. trigonometric functions:

$$\boxed{\begin{aligned} \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{(2n+1)!} \\ \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \end{aligned}}$$

<u>odd</u> powers only

<u>even</u> powers

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \dots + \frac{(iz)^n}{n!} + \dots \qquad i^2 = -1, \; i^3 = -i, \; i^4 = 1 \; \dots$$

$z \to w$: $\boxed{e^{iw} = \cos(w) + i\sin(w)}$ w: real

$(w=\theta)$

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos\theta + i\sin\theta)^n$

$$\boxed{(\cos\theta + i\sin\theta)^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)}$$

DeMoivre's Theorem

(n has to be an integer)

$z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

(i) $z_1 \cdot z_2 = (r_1 r_2)e^{i(\theta_1+\theta_2)} = re^{i\theta}$, $r = r_1 r_2$, $\theta = \theta_1 + \theta_2$

(ii) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = re^{i\theta}$, $r = r_1/r_2$, $\theta = \theta_1 - \theta_2$

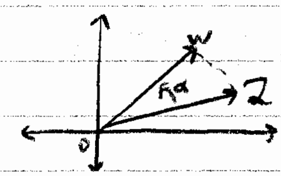
$(z_2 \neq 0)$

ex Show that $|e^{i\alpha}| = 1$ α: real

$e^{i\alpha} = \underbrace{\cos\alpha}_{x} + i\underbrace{\sin\alpha}_{y} \to |e^{i\alpha}| = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$

$|w| = |e^{i\alpha} z| = |e^{i\alpha}||z| = |z|$

rotation of z by an angle α



7. hyperbolic function:

hyperbolic sine: $\sinh z = \dfrac{e^z - e^{-z}}{2}$

hyperbolic cosine: $\cosh z = \dfrac{e^z + e^{-z}}{2}$

$\sinh(iz) = i\sin z$ $\sin(iz) = i\sinh z$

$\cosh(iz) = \cos z$ $\cos(iz) = \cosh z$

$\sin(x+iy) = \sin x \cos iy + \sin iy \cos x = \underbrace{\sin x \cosh y} + i\underbrace{\sinh y \cos x}$