Confidence Intervals for Normal Data

18.05 Spring 2014

Agenda

Today

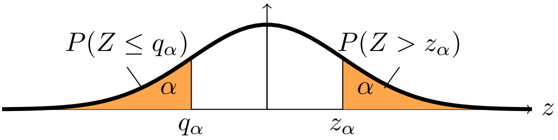
- Review of critical values and quantiles.
- Computing z, t, χ^2 confidence intervals for normal data.
- Conceptual view of confidence intervals.
- Confidence intervals for polling (Bernoulli distributions).

Review of critical values and quantiles

- **Quantile:** left tail $P(X < q_\alpha) = \alpha$
- **Critical value:** right tail $P(X > c_\alpha) = \alpha$

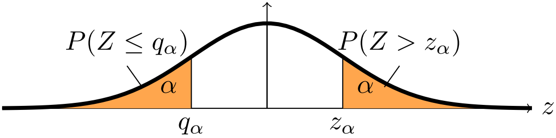
Letters for critical values:

- z_α for $\mathrm{N}(0, 1)$
- t_α for $t(n)$
- c_α, x_α all purpose



q_α and z_α for the standard normal distribution.

Concept question



1. $z_{.025} =$

(a) -1.96 (b) -0.95 (c) 0.95 (d) 1.96 (e) 2.87

2. $-z_{.16} =$

(a) -1.33 (b) -0.99 (c) 0.99 (d) 1.33 (e) 3.52

Solution on next slide.

Solution

1. **answer:** $z_{.025} = 1.96$. By definition $P(Z > z_{.025}) = 0.025$. This is the same as $P(Z \leq z_{.025}) = 0.975$. Either from memory, a table or using the R function `qnorm(.975)` we get the result.

2.**answer:** $-z_{.16} = -0.99$. We recall that $P(|Z| < 1) \approx .68$. Since half the leftover probability is in the right tail we have $P(Z > 1) \approx 0.16$. Thus $z_{.16} \approx 1$, so $-z_{.16} \approx -1$.

Computing confidence intervals from normal data

Suppose the data $x_1, \ldots, x_n$ is drawn from $\mathrm{N}(\mu, \sigma^2)$

Confidence level $= 1 - \alpha$

- z confidence interval for the mean (σ known)

$$\left[\bar{x} - \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}, \ \bar{x} + \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}\right]$$

- t confidence interval for the mean (σ unknown)

$$\left[\bar{x} - \frac{t_{\alpha/2} \cdot s}{\sqrt{n}}, \ \bar{x} + \frac{t_{\alpha/2} \cdot s}{\sqrt{n}}\right]$$

- χ^2 confidence interval for σ^2

$$\left[\frac{n-1}{c_{\alpha/2}} s^2, \ \frac{n-1}{c_{1-\alpha/2}} s^2\right]$$

- t and χ^2 have $n - 1$ degrees of freedom.

z rule of thumb

Suppose $x_1, \ldots, x_n \sim \mathrm{N}(\mu, \sigma^2)$ with σ known.

The rule-of-thumb 95% confidence interval for μ is:

$$\left[\bar{x} - 2\frac{\sigma}{\sqrt{n}}, \quad \bar{x} + 2\frac{\sigma}{\sqrt{n}}\right]$$

A more precise 95% confidence interval for μ is:

$$\left[\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \quad \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

Board question: computing confidence intervals

The data 1, 2, 3, 4 is drawn from $\mathrm{N}(\mu, \sigma^2)$ with μ unknown.

1. Find a 90% z confidence interval for μ, given that $\sigma = 2$.

   For the remaining parts, suppose σ is unknown.

2. Find a 90% t confidence interval for μ.
3. Find a 90% χ^2 confidence interval for σ^2.
4. Find a 90% χ^2 confidence interval for σ.
5. Given a normal sample with $n = 100$, $\overline{x} = 12$, and $s = 5$, find the rule-of-thumb 95% confidence interval for μ.

Solution

$\overline{x} = 2.5, \;\; s^2 = 1.667, \;\; s = 1.29$
$\sigma/\sqrt{n} = 1, \;\; s/\sqrt{n} = 0.645.$
1. $z_{.05} = 1.644$: z confidence interval is

$$2.5 \pm 1.644 \cdot 1 = [0.856, 4.144]$$

2. $t_{.05} = 2.353$ (3 degrees of freedom): t confidence interval is

$$2.5 \pm 2.353 \cdot 0.645 = [0.982, \;\; 4.018]$$

3. $c_{0.05} = 7.1814$, $c_{0.95} = 0.352$ (3 degrees of freedom): χ^2 confidence interval is

$$\left[\frac{3 \cdot 1.667}{7.1814}, \;\; \frac{3 \cdot 1.667}{0.352}\right] = [0.696, \;\; 14.207].$$

4. Take the square root of the interval in 3. $[0.834, \;\; 3.769]$.
5. The rule of thumb is written for z, but with $n = 100$ the $t(99)$ and standard normal distributions are very close, so we can assume that $t_{.025} \approx 2$. Thus the 95% confidence interval is $12 \pm 2 \cdot 5/10 = [11, \;\; 13]$.

Conceptual view of confidence intervals

- Computed from data $\Rightarrow$ **interval statistic**
- 'Estimates' a parameter of interest $\Rightarrow$ **interval estimate**
- Width $=$ measure of precision
- Confidence level $=$ measure of performance
- Confidence intervals are a frequentist method.
  - No need for a prior, only uses likelihood.
  - Frequentists never assign probabilities to unknown parameters:
  - A 95% confidence interval of $[1.2, 3.4]$ for μ **does not mean** that $P(1.2 \le \mu \le 3.4) = 0.95$.
- We will compare with Bayesian probability intervals later.

Applet:
http://mathlets.org/mathlets/confidence-intervals/

Table discussion

The quantities n, c, μ, σ all play a roll in the confidence interval for the mean.

How does the width of a confidence interval for the mean change if:

1. we increase n and leave the others unchanged?
2. we increase c and leave the others unchanged?
3. we increase μ and leave the others unchanged?
4. we increase σ and leave the others unchanged?

(A) it gets wider (B) it gets narrower (C) it stays the same.

Answers

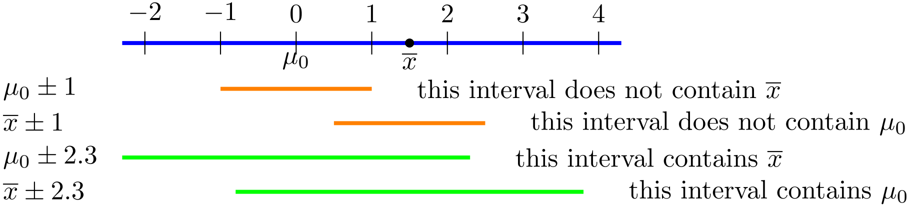
1. Narrower. More data decreases the variance of $\bar{x}$
2. Wider. Greater confidence requires a bigger interval.
3. No change. Changing μ will tend to shift the location of the intervals.
4. Wider. Increasing σ will increase the uncertainty about μ.

Intervals and pivoting

$\overline{x}$: sample mean (statistic)

μ_0: hypothesized mean (not known)

Pivoting: $\overline{x}$ is in the interval $\mu_0 \pm 2.3 \;\Leftrightarrow\; \mu_0$ is in the interval $\overline{x} \pm 2.3$.



Algebra of pivoting:

$$\mu_0 - 2.3 < \overline{x} < \mu_0 + 2.3 \;\Leftrightarrow\; \overline{x} + 2.3 > \mu_0 > \overline{x} - 2.3.$$

Board question: confidence intervals, non-rejection regions

Suppose $x_1, \ldots, x_n \sim \mathrm{N}(\mu, \sigma^2)$ with σ known.

Consider two intervals:

1. The z confidence interval around $\overline{x}$ at confidence level $1 - \alpha$.
2. The z non-rejection region for $H_0 : \mu = \mu_0$ at significance level α.

Compute and sketch these intervals to show that:

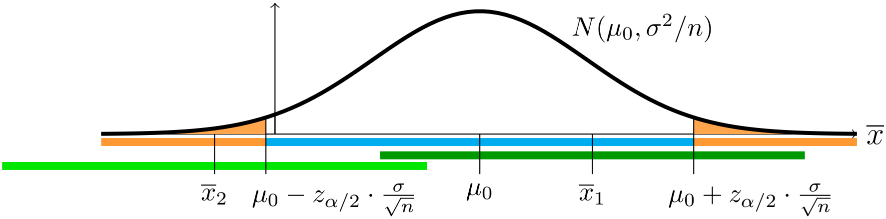
$$\mu_0 \text{ is in the first interval} \quad \Leftrightarrow \quad \overline{x} \text{ is in the second interval.}$$

Solution

Confidence interval: $\overline{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$

Non-rejection region: $\mu_0 \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$

Since the intervals are the same width they either both contain the other's center or neither one does.



Polling: a binomial proportion confidence interval

Data $x_1, \ldots, x_n$ from a Bernoulli(θ) distribution with θ unknown.

A conservative normal† $(1-\alpha)$ confidence interval for θ is given by

$$\left[\bar{x} - \frac{z_{\alpha/2}}{2\sqrt{n}},\ \bar{x} + \frac{z_{\alpha/2}}{2\sqrt{n}}\right].$$

Proof uses the CLT and the observation $\sigma = \sqrt{\theta(1-\theta)} \leq 1/2$.

Political polls often give a margin-of-error of $\pm 1/\sqrt{n}$. This **rule-of-thumb** corresponds to a 95% confidence interval:

$$\left[\bar{x} - \frac{1}{\sqrt{n}},\ \bar{x} + \frac{1}{\sqrt{n}}\right].$$

(The proof is in the class 22 notes.)

Conversely, a margin of error of ± 0.05 means 400 people were polled.

†There are many types of binomial proportion confidence intervals.

http://en.wikipedia.org/wiki/Binomial_proportion_confidence_interval

Board question

For a poll to find the proportion θ of people supporting X we know that a $(1-\alpha)$ confidence interval for θ is given by

$$\left[\bar{x} - \frac{z_{\alpha/2}}{2\sqrt{n}},\ \bar{x} + \frac{z_{\alpha/2}}{2\sqrt{n}}\right].$$

1. How many people would you have to poll to have a margin of error of .01 with 95% confidence? (You can do this in your head.)

2. How many people would you have to poll to have a margin of error of .01 with 80% confidence. (You'll want R or other calculator here.)

3. If $n = 900$, compute the 95% and 80% confidence intervals for θ.
answer: *See next slide.*

<u>answer:</u> 1. Need $1/\sqrt{n} = .01$ So $n = 10000$.

2. $\alpha = .2$, so $z_{\alpha/2} =$ `qnorm(.9)` $= 1.2816$. So we need $\dfrac{z_{\alpha/2}}{2\sqrt{n}} = .01$.

This gives $n = 4106$.

3. 95% interval: $\overline{x} \pm \dfrac{1}{\sqrt{n}} = \overline{x} \pm \dfrac{1}{30} = \overline{x} \pm .0333$

80% interval: $\overline{x} \pm z_{.1}\dfrac{1}{2\sqrt{n}} = \overline{x} \pm 1.2816 \cdot \dfrac{1}{60} = \overline{x} \pm .021$.