18.04 Recitation 6
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1. Let $A = \{z : |z| \le 2\}$, and let $u(x, y)$ be a harmonic function on A. Let $B = \{z : |z| = 2\}$. Express the following in terms of u and B:

1.1. The maximum value of u on A.

Ans: $\max\{u(x, y) : (x, y) \in B\}$

1.2. The minimum value of u on A.

Ans: $\min\{u(x, y) : (x, y) \in B\}$

1.3. The value $u(0, 0)$.

Ans: $\frac{1}{2\pi}\int_0^{2\pi} u(2\cos\theta, 2\sin\theta)d\theta$

2. Let $\Phi(z) = \phi(z) + i\psi(z)$ be an analytic function mapping a region B to another region A. Let $u(x, y)$ be a harmonic function on A.

2.1. Under the assumption that A is simply connected, show that $u(\phi(x, y), \psi(x, y))$ is a harmonic function on B.

Ans: Since u is a harmonic function on the simply connected region A, we can write $u = \mathrm{Re}(f)$ for some analytic function f on A. Then, $u(\phi(x, y), \psi(x, y)) = \mathrm{Re}(f \circ \Phi)$ is harmonic on B, since $f \circ \Phi$ is an analytic function on B.

2.2. Can we drop the assumption that A is simply connected?

Ans: Yes. Let $p(x, y) = u(\phi(x, y), \psi(x, y))$. We will explicitly compute p_{xx} and p_{yy} and show that $p_{xx} + p_{yy} = 0$.

$$\begin{aligned}
p_x &= u_1(\phi, \psi)\phi_x + u_2(\phi, \psi)\psi_x \\
p_{xx} &= u_{11}\phi_x^2 + u_{12}\psi_x\phi_x + u_1\phi_{xx} + u_{21}\phi_x\psi_x + u_{22}\psi_x^2 + u_2\psi_{xx} \\
&= u_{11}\phi_x^2 - 2u_{12}\phi_x\phi_y + u_{22}\phi_y^2 + u_1\phi_{xx} + u_2\psi_{xx} \\
p_y &= u_1(\phi, \psi)\phi_y + u_2(\phi, \psi)\psi_y \\
p_{yy} &= u_{11}\phi_y^2 + u_{12}\psi_y\phi_y + u_1\phi_{yy} + u_{21}\phi_y\psi_y + u_{22}\psi_y^2 + u_2\psi_{yy} \\
&= u_{11}\phi_y^2 + 2u_{12}\phi_x\phi_y + u_{22}\phi_x^2 + u_1\phi_{yy} + u_2\psi_{yy}
\end{aligned}$$

Therefore,

$$\begin{aligned}
p_{xx} + p_{yy} &= (u_{11} + u_{22})(\phi_y^2 + \phi_x^2) + u_1(\phi_{xx} + \phi_{yy}) + u_2(\psi_{xx} + \psi_{yy}) \\
&= 0
\end{aligned}$$

3. Consider the complex potential for the double source: $\Phi(z) = \log(z-1) + \log(z+1) = \log(z^2 - 1)$.

3.1. Find the flow $\boldsymbol{F}$.

Ans: Write $\Phi(x, y) = \phi(x, y) + i\psi(x, y)$. Then

$$\begin{aligned}\phi(x, y) &= \operatorname{Re}(\log(z^2 - 1)) \\ &= \log|(x + iy)^2 - 1| \\ &= \log\sqrt{(x^2 - y^2 - 1)^2 + (2xy)^2} \\ &= \frac{1}{2}\log\left((x^2 - y^2 - 1)^2 + 4x^2y^2\right)\end{aligned}$$

and

$$\boldsymbol{F} = (\phi_x, \phi_y)$$

3.2. Show that on the y-axis, the flow is along the axis.

Ans: Check directly from the above expression for $\phi(x, y)$ that $\phi_x(0, y) = 0$. Another way to do this is to notice that the y-axis is a stream line. Indeed, along the y-axis, the imaginary part of $\log(z^2 - 1) = \log((iy)^2 - 1) = \log(-y^2 - 1)$ is constant.

3.3. What are the stagnation points for this flow?

Ans: $z = 0$. Indeed, $\Phi'(z) = \frac{2z}{z^2-1}$ vanishes only for $z = 0$.

3.4. See the notes for Topic 6 to see the stream lines for this potential and some further discussion.