Topic 6 Notes

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6 Two dimensional hydrodynamics and complex potentials

6.1 Introduction

Laplace's equation and harmonic functions show up in many physical models. As we have just seen, harmonic functions in two dimensions are closely linked with complex analytic functions. In this section we will exploit this connection to look at two dimensional hydrodynamics, i.e. fluid flow.

Since static electric fields and steady state temperature distributions are also harmonic, the ideas and pictures we use can be repurposed to cover these topics as well.

6.2 Velocity fields

Suppose we have water flowing in a region A of the plane. Then at every point (x, y) in A the water has a velocity. In general, this velocity will change with time. We'll let $\mathbf{F}$ stand for the velocity vector field and we can write

$$\mathbf{F}(x, y, t) = (u(x, y, t),\ v(x, y, t)).$$

The arguments (x, y, t) indicate that the velocity depends on these three variables. In general, we will shorten the name to velocity field.

A beautiful and mesmerizing example of a velocity field is at `http://hint.fm/wind/index.html`. This shows the current velocity of the wind at all points in the continental U.S.

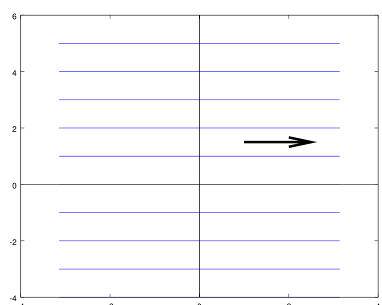
6.3 Stationary flows

If the velocity field is unchanging in time we call the flow a stationary flow. In this case, we can drop t as an argument and write:

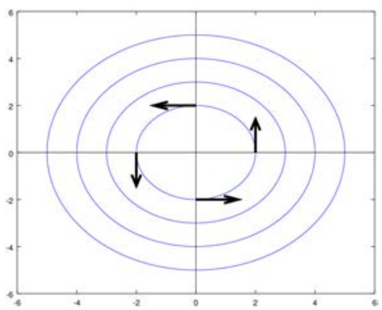
$$\mathbf{F}(x, y) = (u(x, y), v(x, y))$$

Here are a few examples. These pictures show the streamlines from similar figures in topic 5. We've added arrows to indicate the direction of flow.

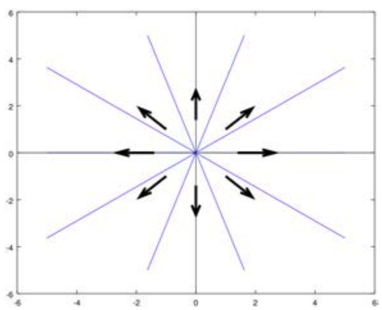
Example 6.1. Uniform flow. $\mathbf{F} = (1, 0)$.



Example 6.2. Eddy (vortex) $\mathbf{F} = (-y/r^2, x/r^2)$



Example 6.3. Source $\mathbf{F} = (x/r^2, y/r^2)$



6.4 Physical assumptions, mathematical consequences

This is a wordy section, so we'll start by listing the mathematical properties that will follow from our assumptions about the velocity field $\mathbf{F} = u + iv$.

(A) $\mathbf{F} = \mathbf{F}(x, y)$ is a function of x,y, but not time t (stationary).

(B) $\operatorname{div}\mathbf{F} = 0$ (divergence free).

(C) $\operatorname{curl}\mathbf{F} = 0$ (curl free).

6.4.1 Physical assumptions

We will make some standard physical assumptions. These don't apply to all flows, but they do apply to a good number of them and they are a good starting point for understanding fluid flow more generally. More important to 18.04, these are the flows that are readily susceptible to complex analysis.

Here are the assumptions about the flow, we'll discuss them further below:

(A) The flow is stationary.

(B) The flow is incompressible.

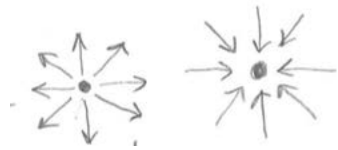
(C) The flow is irrotational.

We have already discussed stationarity in Section 6.3, so let's now discuss the other two properties.

(B) **Incompressibility.** We will assume throughout that the fluid is incompressible. This means that the density of the fluid is constant across the domain. Mathematically this says that the velocity field $\mathbf{F}$ must be divergence free, i.e. for $\mathbf{F} = (u, v)$:

$$\text{div}\mathbf{F} \equiv \nabla \cdot \mathbf{F} = u_x + v_y = 0.$$

To understand this, recall that the divergence measures the infinitesimal flux of the field. If the flux is not zero at a point (x_0, y_0) then near that point the field looks like



Left: Divergent field: $\text{div}\mathbf{F} > 0$, right: Convergent field: $\text{div}\mathbf{F} < 0$

If the field is diverging or converging then the density must be changing! That is, the flow is not incompressible.

As a fluid flow the left hand picture represents a source and the right represents a sink. In electrostatics where $\mathbf{F}$ expresses the electric field, the left hand picture is the field of a positive charge density and the right is that of a negative charge density.

If you prefer a non-infinitesimal explanation, we can recall Green's theorem in flux form. It says that for a simple closed curve C and a field $\mathbf{F} = (u, v)$, differentiable on and inside C, the flux of $\mathbf{F}$ through C satisfies

$$\text{Flux of } \mathbf{F} \text{ across } C \ = \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int\int_R \text{div}\mathbf{F} \, dx \, dy,$$

where R is the region inside C. Now, suppose that $\text{div}\mathbf{F}(x_0, y_0) > 0$, then $\text{div}\mathbf{F}(x, y) > 0$ for all (x, y) close to (x_0, y_0). So, choose a small curve C around (x_0, y_0) such that $\text{div}\mathbf{F} > 0$ on and inside C. By Green's theorem

$$\text{Flux of } \mathbf{F} \text{ through } C \ = \int\int_R \text{div}\mathbf{F} \, dx \, dy > 0.$$

Clearly, if there is a net flux out of the region the density is decreasing and the flow is not incompressible. The same argument would hold if $\text{div}\mathbf{F}(x_0, y_0) < 0$. We conclude that incompressible is equivalent to divergence free.

(C) **Irrotational flow.** We will assume that the fluid is irrotational. This means that the there are no infinitesimal vortices in A. Mathematically this says that the velocity field $\mathbf{F}$ must be curl free, i.e. for $\mathbf{F} = (u, v)$:

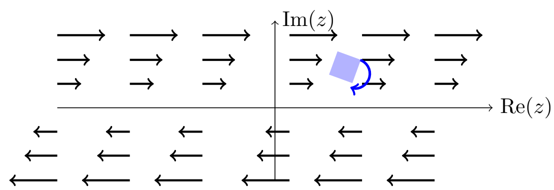
$$\text{curl}\mathbf{F} \equiv \nabla \times \mathbf{F} = v_x - u_y = 0.$$

To understand this, recall that the curl measures the infinitesimal rotation of the field. Physically this means that a small paddle placed in the flow will not spin as it moves with the flow.

6.4.2 Examples

Example 6.4. The eddy is irrotational! The eddy from Example 6.2 is irrotational. The vortex at the origin is not in $A = \mathbf{C} - \{0\}$ and you can easily check that $\text{curl}\mathbf{F} = 0$ everywhere in A. This is not physically impossible: if you placed a small paddle wheel in the flow it would travel around the origin without spinning!

Example 6.5. Shearing flows are rotational. Here's an example of a vector field that has rotation, though not necessarily swirling.



Shearing flow: box turns because current is faster at the top.

The field $\mathbf{F} = (ay, 0)$ is horizontal, but curl$\mathbf{F} = -a \neq 0$. Because the top moves faster than the bottom it will rotate a square parcel of fluid. The minus sign tells you the parcel will rotate clockwise! This is called a shearing flow. The water at one level will be sheared away from the level above it.

6.4.3 Summary

(A) Stationary: $\mathbf{F}$ depends on x, y, but not t, i.e.,

$$\mathbf{F}(x, y) = (u(x, y), v(x, y)).$$

(B) Incompressible: divergence free:

$$\text{div}\mathbf{F} = u_x + v_y = 0, \quad \text{i.e.} \quad u_x = -v_y.$$

(C) Irrotational: curl free:

$$\text{curl}\mathbf{F} = v_x - u_y = 0, \qquad \text{i.e.,} \qquad u_y = v_x.$$

For future reference we put the last two equalities in a numbered equation:

$$u_x = -v_y \qquad \text{and} \qquad u_y = v_x \tag{1}$$

These look almost like the Cauchy-Riemann equations (with sign differences)!

6.5 Complex potentials

There are different ways to do this. We'll start by seeing that every complex analytic function leads to an irrotational, incompressible flow. Then we'll go backwards and see that all such flows lead to an analytic function. We will learn to call the analytic function the complex potential of the flow.

Annoyingly, we are going to have to switch notation. Because u and v are already taken by the vector field $\mathbf{F}$, we will call our complex potential

$$\Phi = \phi + i\psi.$$

6.5.1 Analytic functions give us incompressible, irrotational flows

Let $\Phi(z)$ be an analytic function on a region A. For $z = x + iy$ we write

$$\Phi(z) = \phi(x, y) + i\psi(x, y).$$

From this we can define a vector field

$$\mathbf{F} = \nabla\phi = (\phi_x, \phi_y) =: (u, v),$$

here we mean that u and v are defined by ϕ_x and ϕ_y.

From our work on analytic and harmonic functions we can make a list of properties of these functions.

1. ϕ and ψ are both harmonic.
2. The level curves of ϕ and ψ are orthogonal.
3. $\Phi' = \phi_x - i\phi_y$.
4. $\mathbf{F}$ is divergence and curl free (proof just below). That is, the analytic function Φ has given us an incompressible, irrotational vector field $\mathbf{F}$.

It is standard terminology to call ϕ a potential function for the vector field $\mathbf{F}$. We will also call Φ a complex potential function for $\mathbf{F}$. The function ψ will be called the stream function of $\mathbf{F}$ (the name will be explained soon). The function Φ' will be called the complex velocity.

Proof. ($\mathbf{F}$ is curl and divergence free.) This is an easy consequence of the definition. We find

$$\text{curl}\mathbf{F} = v_x - u_y = \phi_{yx} - \phi_{xy} = 0$$

$$\text{div}\mathbf{F} = u_x + v_y = \phi_{xx} + \phi_{yy} = 0 \text{ (since } \phi \text{ is harmonic).}$$

We'll postpone examples until after deriving the complex potential from the flow.

6.5.2 Incompressible, irrotational flows always have complex potential functions

For technical reasons we need to add the assumption that A is simply connected. This is not usually a problem because we often work locally in a disk around a point (x_0, y_0).

Theorem. Assume $\mathbf{F} = (u, v)$ is an incompressible, irrotational field on a simply connected region A. Then there is an analytic function Φ which is a complex potential function for $\mathbf{F}$.

Proof. We have done all the heavy lifting for this in previous topics. The key is to use the property $\Phi' = u - iv$ to guess Φ'. Working carefully we define

$$g(z) = u - iv$$

Step 1: Show that g is analytic. Keeping the signs straight, the Cauchy Riemann equations are

$$u_x = (-v)_y \text{ and } u_y = -(-v)_x = v_x.$$

But, these are exactly the equations in Equation 1. Thus $g(z)$ is analytic.

Step 2: Since A is simply connected, Cauchy's theorem says that $g(z)$ has an antiderivative on A. We call the antiderivative $\Phi(z)$.

Step 3: Show that $\Phi(z)$ is a complex potential function for $\mathbf{F}$. This means we have to show that if we write $\Phi = \phi + i\psi$, then $\mathbf{F} = \nabla\phi$. To do this we just unwind the definitions.

$$\begin{aligned} \Phi' &= \phi_x - i\phi_y && \text{(standard formula for } \Phi'\text{)} \\ \Phi' &= g = u - iv && \text{(definition of } \Phi \text{ and } g\text{)} \end{aligned}$$

Comparing these equations we get

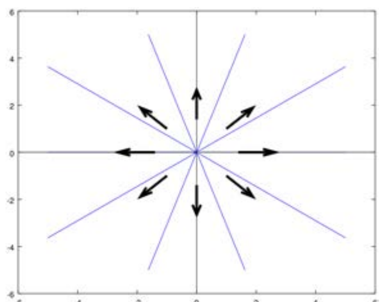
$$\phi_x = u, \qquad \phi_y = v.$$

But this says precisely that $\nabla\phi = \mathbf{F}$. QED

Example 6.6. Source fields. The vector field

$$\mathbf{F} = a\left(\frac{x}{r^2}, \frac{y}{r^2}\right)$$

models a source pushing out water or the 2D electric field of a positive charge at the origin. (If you prefer a 3D model, it is the field of an infinite wire with uniform charge density along the z-axis.) Show that $\mathbf{F}$ is curl-free and divergence-free and find its complex potential.



We could compute directly that this is curl-free and divergence-free away from 0. An alternative method is to look for a complex potential Φ. If we can find one then this will show $\mathbf{F}$ is curl and divergence free and find ϕ and ψ all at once. If there is no such Φ then we'll know that $\mathbf{F}$ is not both curl and divergence free.

One standard method is to use the formula for Φ':

$$\Phi' = u - iv = a\frac{(x - iy)}{r^2} = a\frac{\overline{z}}{(\overline{z}z)} = \frac{a}{z}.$$

This is analytic and we have

$$\Phi(z) = a\log(z).$$

6.6 Stream functions

In everything we did above poor old ψ just tagged along as the harmonic conjugate of the potential function ϕ. Let's turn our attention to it and see why it's called the stream function.

Theorem. Suppose that

$$\Phi = \phi + i\psi$$

is the complex potential for a velocity field $\mathbf{F}$. Then the fluid flows along the level curves of ψ. That is, the $\mathbf{F}$ is everywhere tangent to the level curves of ψ. The level curves of ψ are called streamlines and ψ is called the stream function.

Proof. Again we have already done most of the heavy lifting to prove this. Since $\mathbf{F}$ is the velocity of the flow at each point, the flow is always tangent to $\mathbf{F}$. You also need to remember that $\nabla\phi$ is perpendicular to the level curves of ϕ. So we have:

1. The flow is parallel to $\mathbf{F}$.
2. $\mathbf{F} = \nabla\phi$, so the flow is orthogonal to the level curves of ϕ.
3. Since ϕ and ψ are harmonic conjugates, the level curves of ψ are orthogonal to the level curves of ϕ.

Combining 2 and 3 we see that the flow must be along the level curves of ψ.

6.6.1 Examples

We'll illustrate the streamlines in a series of examples that start by defining the complex potential for a vector field.

Example 6.7. Uniform flow. Let

$$\Phi(z) = z.$$

Find $\mathbf{F}$ and draw a plot of the streamlines. Indicate the direction of the flow.

Solution: Write

$$\Phi = x + iy.$$

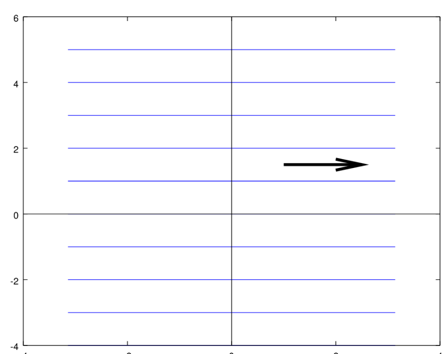
So

$$\phi = x \qquad \text{and} \qquad \mathbf{F} = \nabla\phi = (1, 0),$$

which says the flow has uniform velocity and points to the right. We also have

$$\psi = y,$$

so the streamlines are the horizontal lines $y =$ constant.



Uniform flow to the right.

Note that another way to see that the flow is to the right is to check the direction that the potential ϕ increases. The Topic 5 notes show pictures of this complex potential which show both the streamlines and the equipotential lines.

Example 6.8. Linear source. Let

$$\Phi(z) = \log(z).$$

Find **F** and draw a plot of the streamlines. Indicate the direction of the flow.

Solution: Write

$$\Phi = \log(r) + i\theta.$$

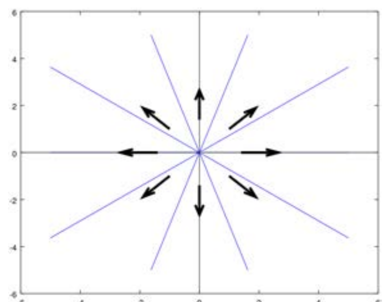
So

$$\phi = \log(r) \quad \text{and} \quad \mathbf{F} = \nabla\phi = (x/r^2, y/r^2),$$

which says the flow is radial and decreases in speed as it gets farther from the origin. The field is not defined at $z = 0$. We also have

$$\psi = \theta,$$

so the streamlines are rays from the origin.



Linear source: radial flow from the origin.

6.6.2 Stagnation points

A stagnation point is one where the velocity field is 0.

Stagnation points. If Φ is the complex potential for a field **F** then the stagnation points $\mathbf{F} = 0$ are exactly the points z where $\Phi'(z) = 0$.

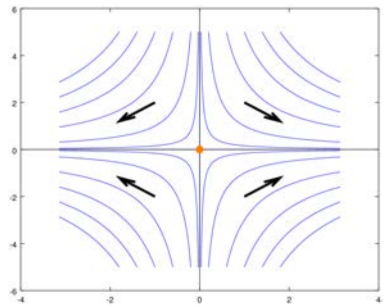
Proof. This is clear since $\mathbf{F} = (\phi_x, \phi_y)$ and $\Phi' = \phi_x - i\phi_y$.

Example 6.9. Stagnation points. Draw the streamlines and identify the stagnation points for the potential $\Phi(z) = z^2$.

Solution: (We drew the level curves for this in Topic 5.) We have

$$\Phi = (x^2 - y^2) + i2xy.$$

So the streamlines are the hyperbolas: $2xy =$ constant. Since $\phi = x^2 - y^2$ increases as $|x|$ increases and decreases as $|y|$ increases, the arrows, which point in the direction of increasing ϕ, are as shown on the figure below.



Stagnation flow: stagnation point at $z = 0$.

The stagnation points are the zeros of

$$\Phi'(z) = 2z,$$

i.e. the only stagnation point is at the $z = 0$.

Note. The stagnation points are what we called the critical points of a vector field in 18.03.

6.7 More examples with pretty pictures

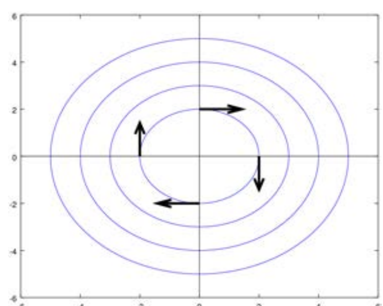
Example 6.10. Linear vortex. Analyze the flow with complex potential function

$$\Phi(z) = i\log(z).$$

Solution: Multiplying by i switches the real and imaginary parts of $\log(z)$ (with a sign change). We have

$$\Phi = -\theta + i\log(r).$$

The stream lines are the curves $\log(r) = \text{constant}$, i.e. circles with center at $z = 0$. The flow is clockwise because the potential $\phi = -\theta$ increases in the clockwise direction.



Linear vortex.

This flow is called a linear vortex. We can find $\mathbf{F}$ using Φ'.

$$\Phi' = \frac{i}{z} = \frac{y}{r^2} + i\frac{x}{r^2} = \phi_x - i\phi_y.$$

So

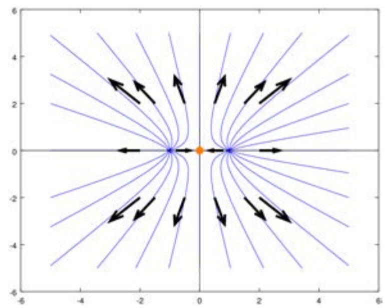
$$\mathbf{F} = (\phi_x, \phi_y) = (y/r^2, -x/r^2).$$

(By now this should be a familiar vector field.) There are no stagnation points, but there is a singularity at the origin.

Example 6.11. Double source. Analyze the flow with complex potential function

$$\Phi(z) = \log(z-1) + \log(z+1).$$

Solution: This is a flow with linear sources at ± 1. We used Octave to plot the level curves of $\psi = \text{Im}(\Phi)$.



Two sources.

We can analyze this flow further as follows.

- Near each source the flow looks like a linear source.
- On the y-axis the flow is along the axis. That is, the y-axis is a streamline. It's worth seeing three different ways of arriving at this conclusion.

Reason 1: By symmetry of vector fields associated with each linear source, the x components cancel and the combined field points along the y-axis.

Reason 2: We can write

$$\Phi(z) = \log(z-1) + \log(z+1) = \log((z-1)(z+1)) = \log(z^2-1).$$

So

$$\Phi'(z) = \frac{2z}{z^2-1}.$$

On the imaginary axis

$$\Phi'(iy) = \frac{2iy}{-y^2-1}.$$

Thus,

$$\mathbf{F} = \left(0, \frac{2y}{y^2+1}\right)$$

which is along the axis.

Reason 3: On the imaginary axis $\Phi(iy) = \log(-y^2-1)$. Since this has constant imaginary part, the axis is a streamline.

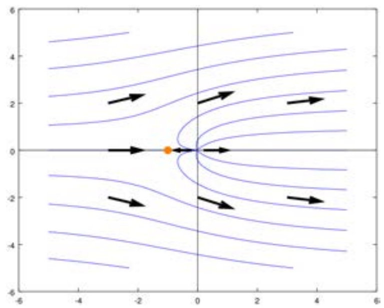
Because of the branch cut for $\log(z)$ we should probably be a little more careful here. First note that the vector field $\mathbf{F}$ comes from $\Phi' = 2z/(z^2-1)$, which doesn't have a branch cut. So we shouldn't really have a problem. Now, as z approaches the y-axis from one side or the other, the argument of $\log(z^2 - 1)$ approaches either π or $-\pi$. That is, as such limits, the imaginary part is constant. So the streamline on the y-axis is the limit case of streamlines near the axis.

Since $\Phi'(z) = 0$ when $z = 0$, the origin is a stagnation point. This is where the fields from the two sources exactly cancel each other.

Example 6.12. A source in uniform flow. Consider the flow with complex potential

$$\Phi(z) = z + \frac{Q}{2\pi} \log(z).$$

This is a combination of uniform flow to the right and a source at the origin. The figure below was drawn using Octave. It shows that the flow looks like a source near the origin. Farther away from the origin the flow stops being radial and is pushed to the right by the uniform flow.



A source in uniform flow.

Since the components of Φ' and $\mathbf{F}$ are the same except for signs, we can understand the flow by considering

$$\Phi'(z) = 1 + \frac{Q}{2\pi z}.$$

Near $z = 0$ the singularity of $1/z$ is most important and

$$\Phi' \approx \frac{Q}{2\pi z}.$$

So, the vector field looks a linear source. Far away from the origin the $1/z$ term is small and $\Phi'(z) \approx 1$, so the field looks like uniform flow.

Setting $\Phi'(z) = 0$ we find one stagnation point

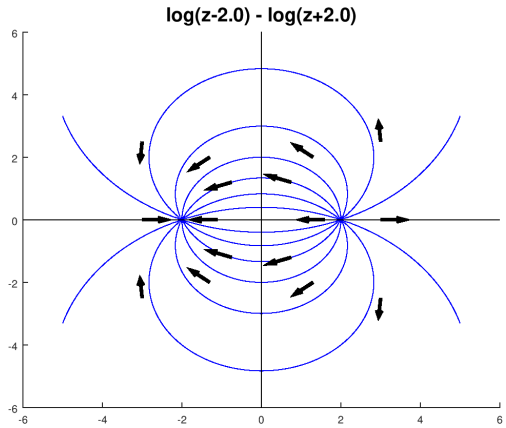
$$z = -\frac{Q}{2\pi}.$$

It is the point on the x-axis where the flow from the source exactly balances that from the uniform flow. For bigger values of Q the source pushes fluid farther out before being overwhelmed by the uniform flow. That is why Q is called the source strength.

Example 6.13. Source + sink. Consider the flow with complex potential

$$\Phi(z) = \log(z - 2) - \log(z + 2).$$

This is a combination of source $(\log(z - 2))$ at $z = 2$ and a sink $(-\log(z + 2))$ at $z = -2$.



A source plus a sink.

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18.04 Complex Variables with Applications
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