Topic 5 Notes

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5 Introduction to harmonic functions

5.1 Introduction

Harmonic functions appear regularly and play a fundamental role in math, physics and engineering. In this topic we'll learn the definition, some key properties and their tight connection to complex analysis. The key connection to 18.04 is that both the real and imaginary parts of analytic functions are harmonic. We will see that this is a simple consequence of the Cauchy-Riemann equations. In the next topic we will look at some applications to hydrodynamics.

5.2 Harmonic functions

We start by defining harmonic functions and looking at some of their properties.

Definition 5.1. A function $u(x, y)$ is called harmonic if it is twice continuously differentiable and satisfies the following partial differential equation:

$$\nabla^2 u = u_{xx} + u_{yy} = 0. \tag{1}$$

Equation 1 is called Laplace's equation. So a function is harmonic if it satisfies Laplace's equation. The operator ∇^2 is called the Laplacian and $\nabla^2 u$ is called the Laplacian of u.

5.3 Del notation

Here's a quick reminder on the use of the notation $\boldsymbol{\nabla}$. For a function $u(x, y)$ and a vector field $\mathbf{F}(x, y) = (u, v)$, we have

(i) $\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

(ii) $\operatorname{grad} u = \boldsymbol{\nabla} u = (u_x, u_y)$

(iii) $\operatorname{curl} \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} = (v_x - u_y)$

(iv) $\operatorname{div} \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F} = u_x + v_y$

(v) $\operatorname{div} \operatorname{grad} u = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u = \nabla^2 u = u_{xx} + u_{yy}$

(vi) $\operatorname{curl} \operatorname{grad} u = \boldsymbol{\nabla} \times \boldsymbol{\nabla}\, u = 0$

(vii) $\operatorname{div} \operatorname{curl} \mathbf{F} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{F} = 0.$

5.3.1 Analytic functions have harmonic pieces

The connection between analytic and harmonic functions is very strong. In many respects it mirrors the connection between e^z and sine and cosine.

Let $z = x + iy$ and write $f(z) = u(x, y) + iv(x, y)$.

Theorem 5.2. If $f(z) = u(x, y) + iv(x, y)$ is analytic on a region A then both u and v are harmonic functions on A.

Proof. This is a simple consequence of the Cauchy-Riemann equations. Since $u_x = v_y$ we have

$$u_{xx} = v_{yx}.$$

Likewise, $u_y = -v_x$ implies

$$u_{yy} = -v_{xy}.$$

Since $v_{xy} = v_{yx}$ we have

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

Therefore u is harmonic. We can handle v similarly. □

Note. Since we know an analytic function is infinitely differentiable we know u and v have the required two continuous partial derivatives. This also ensures that the mixed partials agree, i.e. $v_{xy} = v_{yx}$.

To complete the tight connection between analytic and harmonic functions we show that any harmonic function is the real part of an analytic function.

Theorem 5.3. If $u(x, y)$ is harmonic on a simply connected region A, then u is the real part of an analytic function $f(z) = u(x, y) + iv(x, y)$.

Proof. This is similar to our proof that an analytic function has an antiderivative. First we come up with a candidate for $f(z)$ and then show it has the properties we need. Here are the details broken down into steps 1-4.

1. Find a candidate, call it $g(z)$, for $f'(z)$:

   If we had an analytic f with $f = u + iv$, then Cauchy-Riemann says that $f' = u_x - iu_y$. So, let's define

   $$g = u_x - iu_y.$$

   This is our candidate for f'.

2. Show that $g(z)$ is analytic:

   Write $g = \phi + i\psi$, where $\phi = u_x$ and $\psi = -u_y$. Checking the Cauchy-Riemann equations we have

   $$\begin{bmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{bmatrix} = \begin{bmatrix} u_{xx} & u_{xy} \\ -u_{yx} & -u_{yy} \end{bmatrix}$$

   Since u is harmonic we know $u_{xx} = -u_{yy}$, so $\phi_x = \psi_y$. It is clear that $\phi_y = -\psi_x$. Thus g satisfies the Cauchy-Riemann equations, so it is analytic.

3. Let f be an antiderivative of g:

Since A is simply connected our statement of Cauchy's theorem guarantees that $g(z)$ has an antiderivative in A. We'll need to fuss a little to get the constant of integration exactly right. So, pick a base point z_0 in A. Define the antiderivative of $g(z)$ by

$$f(z) = \int_{z_0}^{z} g(z)\,dz + u(x_0, y_0).$$

(Again, by Cauchy's theorem this integral can be along any path in A from z_0 to z.)

4. Show that the real part of f is u.

   Let's write $f = U + iV$. So, $f'(z) = U_x - iU_y$. By construction

   $$f'(z) = g(z) = u_x - iu_y.$$

   This means the first partials of U and u are the same, so U and u differ by at most a constant. However, also by construction,

   $$f(z_0) = u(x_0, y_0) = U(x_0, y_0) + iV(x_0, y_0),$$

   So, $U(x_0, y_0) = u(x_0, y_0)$ (and $V(x_0, y_0) = 0$). Since they agree at one point we must have $U = u$, i.e. the real part of f is u as we wanted to prove.

Important corollary. u is infinitely differentiable.

Proof. By definition we only require a harmonic function u to have continuous second partials. Since the analytic f is infinitely differentiable, we have shown that so is u!

5.3.2 Harmonic conjugates

Definition. If u and v are the real and imaginary parts of an analytic function, then we say u and v are harmonic conjugates.

Note. If $f(z) = u + iv$ is analytic then so is $if(z) = -v + iu$. So, if u and v are harmonic conjugates and so are u and $-v$.

5.4 A second proof that u and v are harmonic

This fact is important enough that we will give a second proof using Cauchy's integral formula. One benefit of this proof is that it reminds us that Cauchy's integral formula can transfer a general question on analytic functions to a question about the function $1/z$. We start with an easy to derive fact.

Fact. The real and imaginary parts of $f(z) = 1/z$ are harmonic away from the origin. Likewise for

$$g(z) = f(z - a) = \frac{1}{z - a}$$

away from the point $z = a$.

Proof. We have

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

It is a simple matter to apply the Laplacian and see that you get 0. We'll leave the algebra to you! The statement about $g(z)$ follows in either exactly the same way, or by noting that the Laplacian is translation invariant.

Second proof that f analytic implies u and v are harmonic. We are proving that if $f = u + iv$ is analytic then u and v are harmonic. So, suppose f is analytic at the point z_0. This means there is a disk of some radius, say r, around z_0 where f is analytic. Cauchy's formula says

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w - z}\, dw,$$

where C_r is the circle $|w - z_0| = r$ and z is in the disk $|z - z_0| < r$.

Now, since the real and imaginary parts of $1/(w - z)$ are harmonic, the same must be true of the integral, which is limit of linear combinations of such functions. Since the circle is finite and f is continuous, interchanging the order of integration and differentiation is not a problem.

5.5 Maximum principle and mean value property

These are similar to the corresponding properties of analytic functions. Indeed, we deduce them from those corresponding properties.

Theorem. (Mean value property) If u is a harmonic function then u satisfies the mean value property. That is, suppose u is harmonic on and inside a circle of radius r centered at $z_0 = x_0 + iy_0$ then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

Proof. Let $f = u + iv$ be an analytic function with u as its real part. The mean value property for f says

$$\begin{aligned} u(x_0, y_0) + iv(x_0, y_0) = f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) + iv(z_0 + re^{i\theta})\, d\theta \end{aligned}$$

Looking at the real parts of this equation proves the theorem.

Theorem. (Maximum principle) Suppose $u(x, y)$ is harmonic on a open region A.

(i) Suppose z_0 is in A. If u has a relative maximum or minimum at z_0 then u is constant on a disk centered at z_0.

(ii) If A is bounded and connected and u is continuous on the boundary of A then the absolute maximum and absolute minimum of u occur on the boundary.

Proof. The proof for maxima is identical to the one for the maximum modulus principle. The proof for minima comes by looking at the maxima of $-u$.

Note. For analytic functions we only talked about maxima because we had to use the modulus in order to have real values. Since $|-f| = |f|$ we couldn't use the trick of turning minima into maxima by using a minus sign.

5.6 Orthogonality of curves

An important property of harmonic conjugates u and v is that their level curves are orthogonal. We start by showing their gradients are orthogonal.

Lemma 5.4. Let $z = x + iy$ and suppose that $f(z) = u(x, y) + iv(x, y)$ is analytic. Then the dot product of their gradients is 0, i.e.

$$\nabla u \cdot \nabla v = 0.$$

Proof. The proof is an easy application of the Cauchy-Riemann equations.

$$\nabla u \cdot \nabla v = (u_x, u_y) \cdot (v_x, v_y) = u_x v_x + u_y v_y = v_y v_x - v_x v_y = 0$$

In the last step we used the Cauchy-Riemann equations to substitute v_y for u_x and $-v_x$ for u_y. □

The lemma holds whether or not the gradients are 0. To guarantee that the level curves are smooth the next theorem requires that $f'(z) \neq 0$.

Theorem. Let $z = x + iy$ and suppose that

$$f(z) = u(x, y) + iv(x, y)$$

is analytic. If $f'(z) \neq 0$ then the level curve of u through (x, y) is orthogonal to the level curve v through (x, y).

Proof. The technical requirement that $f'(z) \neq 0$ is needed to be sure that the level curves are smooth. We need smoothness so that it even makes sense to ask if the curves are orthogonal. We'll discuss this below. Assuming the curves are smooth the proof of the theorem is trivial: We know from 18.02 that the gradient ∇u is orthogonal to the level curves of u and the same is true for ∇v and the level curves of v. Since, by Lemma 5.4, the gradients are orthogonal this implies the curves are orthogonal.

Finally, we show that $f'(z) \neq 0$ means the curves are smooth. First note that

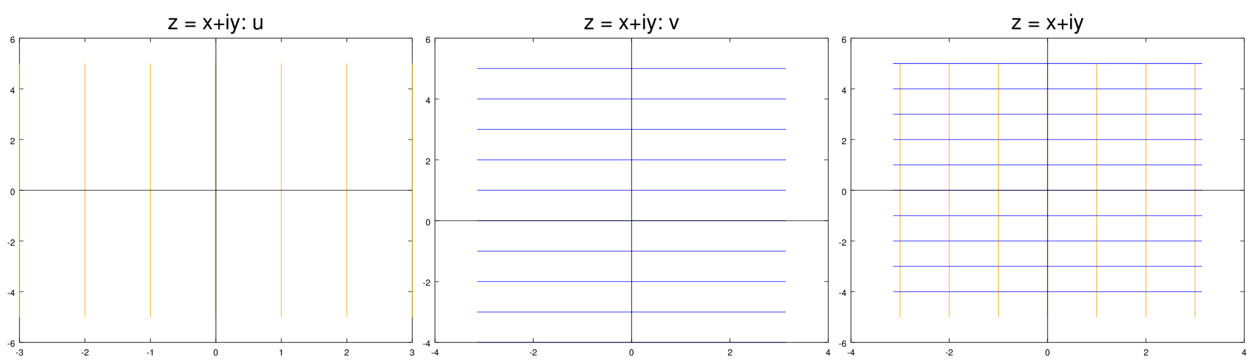
$$f'(z) = u_x(x, y) - iu_y(x, y) = v_y(x, y) + iv_x(x, y).$$

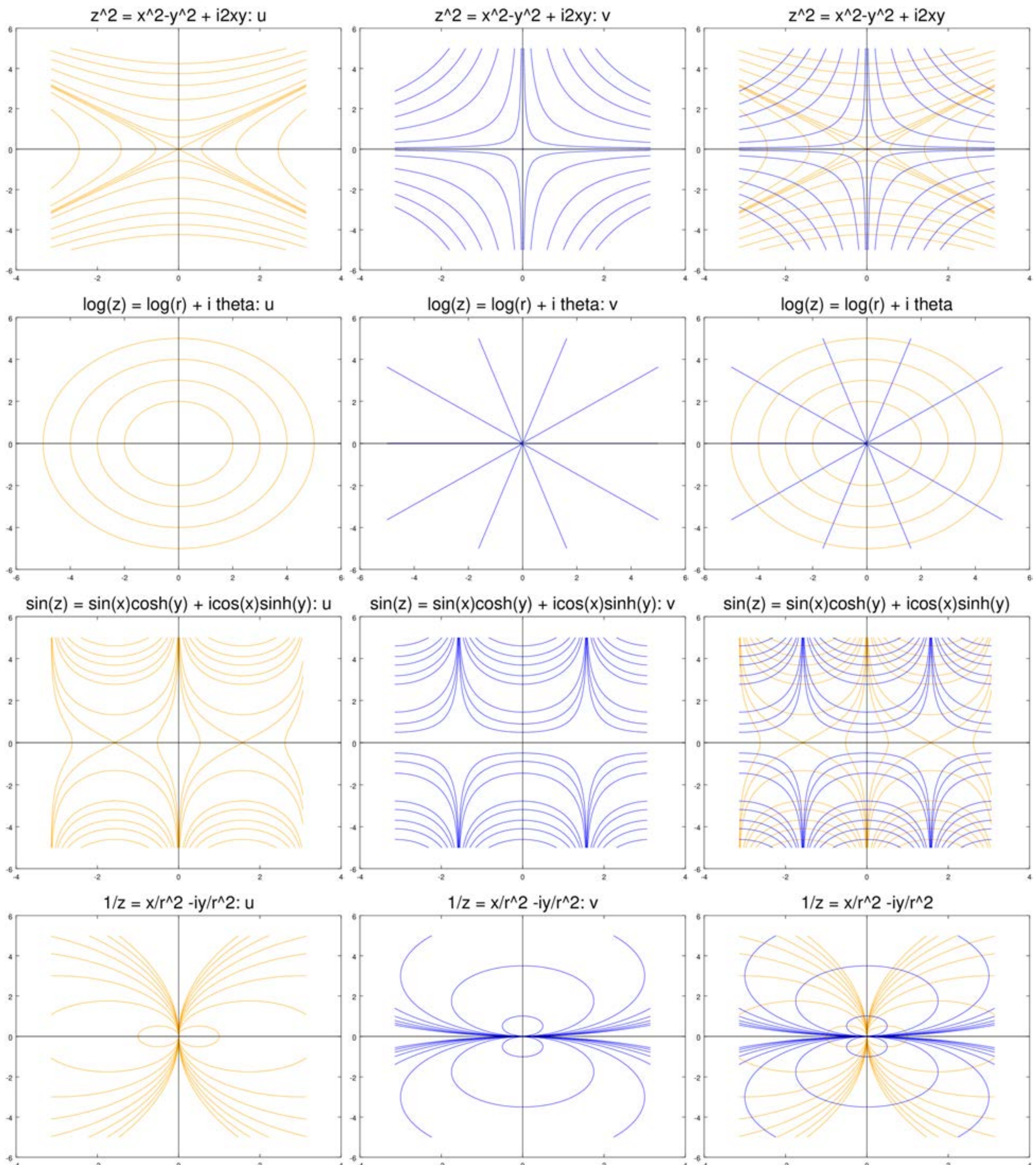
Now, since $f'(z) \neq 0$ we know that

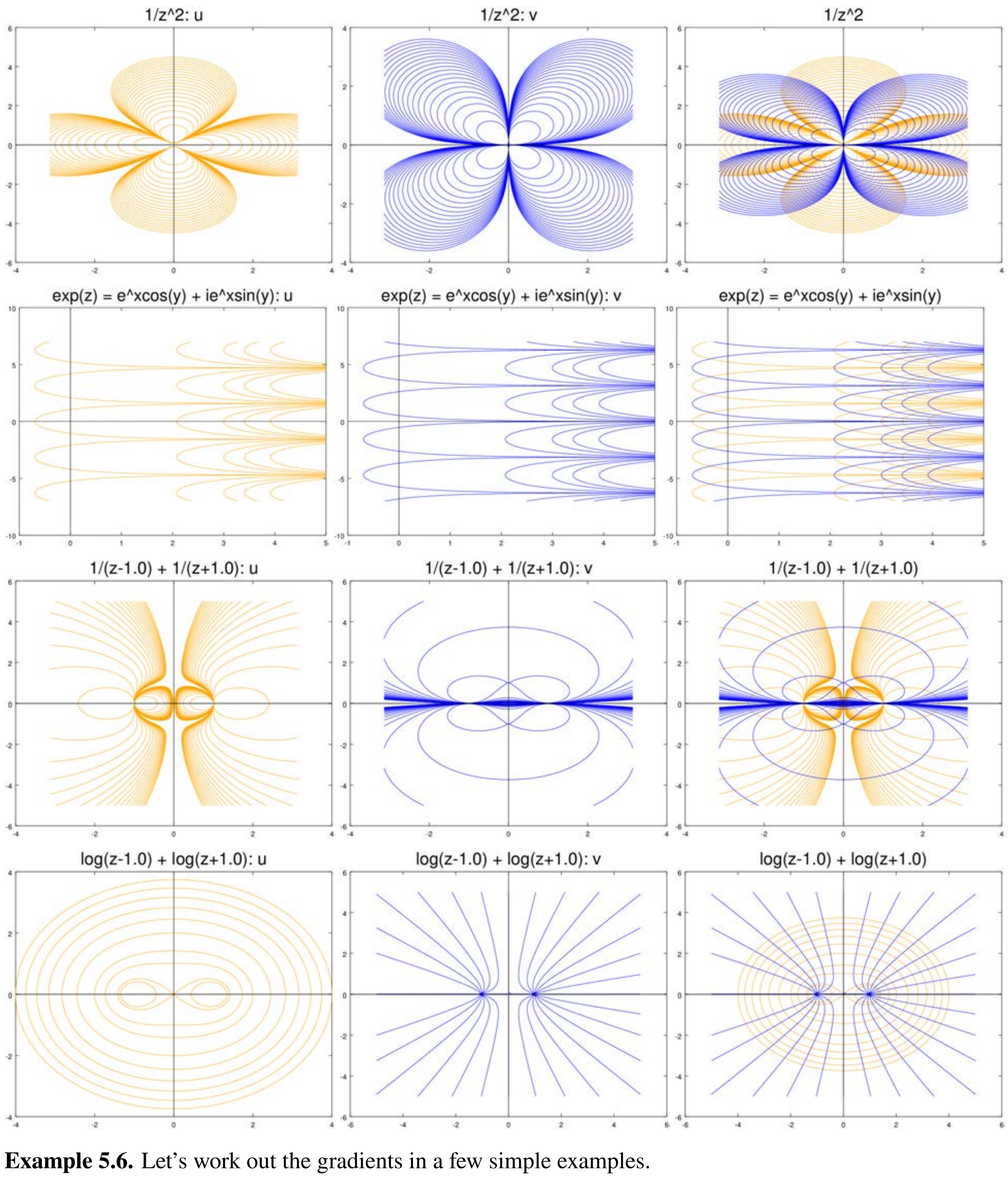
$$\nabla u = (u_x, u_y) \neq 0.$$

Likewise, $\nabla v \neq 0$. Thus, the gradients are not zero and the level curves must be smooth.

Example 5.5. The figures below show level curves of u and v for a number of functions. In all cases, the level curves of u are in orange and those of v are in blue. For each case we show the level curves separately and then overlayed on each other.




z^2 = x^2-y^2 + i2xy: u
z^2 = x^2-y^2 + i2xy: v
z^2 = x^2-y^2 + i2xy
log(z) = log(r) + i theta: u
log(z) = log(r) + i theta: v
log(z) = log(r) + i theta
sin(z) = sin(x)cosh(y) + icos(x)sinh(y): u
sin(z) = sin(x)cosh(y) + icos(x)sinh(y): v
sin(z) = sin(x)cosh(y) + icos(x)sinh(y)
1/z = x/r^2 -iy/r^2: u
1/z = x/r^2 -iy/r^2: v
1/z = x/r^2 -iy/r^2



Example 5.6. Let's work out the gradients in a few simple examples.

(i) Let

$$f(z) = z^2 = (x^2 - y^2) + i2xy,$$

So

$$\nabla u = (2x, -2y) \quad \text{and} \quad \nabla v = (2y, 2x).$$

It's trivial to check that $\nabla u \cdot \nabla v = 0$, so they are orthogonal.

(ii) Let

$$f(z) = \frac{1}{z} = \frac{x}{r^2} - i\frac{y}{r^2}.$$

So, it's easy to compute

$$\nabla u = \left(\frac{y^2 - x^2}{r^4}, \frac{-2xy}{r^4}\right) \qquad \text{and} \qquad \nabla v = \left(\frac{2xy}{r^4}, \frac{y^2 - x^2}{r^4}\right).$$

Again it's trivial to check that $\nabla u \cdot \nabla v = 0$, so they are orthogonal.

Example 5.7. (Degenerate points: $f'(z) = 0$.) Consider

$$f(z) = z^2.$$

From the previous example we have

$$u(x, y) = x^2 - y^2, \qquad v(x, y) = 2xy, \qquad \nabla u = (2x, -2y), \qquad \nabla v = (2y, 2x).$$

At $z = 0$, the gradients are both 0 so the theorem on orthogonality doesn't apply.

Let's look at the level curves through the origin. The level curve (really the 'level set') for

$$u = x^2 - y^2 = 0$$

is the pair of lines $y = \pm x$. At the origin this is not a smooth curve.

Look at the figures for z^2 above. It does appear that away from the origin the level curves of u intersect the lines where $v = 0$ at right angles. The same is true for the level curves of v and the lines where $u = 0$. You can see the degeneracy forming at the origin: as the level curves head towards 0 they get pointier and more right angled. So the level curve $u = 0$ is more properly thought of as four right angles. The level curve of $u = 0$, not knowing which leg of $v = 0$ to intersect orthogonally takes the average and comes into the origin at 45°.