Topic 10 Notes

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10 Conformal transformations

10.1 Introduction

In this topic we will look at the geometric notion of conformal maps. It will turn out that analytic functions are automatically conformal. Once we have understood the general notion, we will look at a specific family of conformal maps called fractional linear transformations and, in particular at their geometric properties. As an application we will use fractional linear transformations to solve the Dirichlet problem for harmonic functions on the unit disk with specified values on the unit circle. At the end we will return to some questions of fluid flow.

10.2 Geometric definition of conformal mappings

We start with a somewhat hand-wavy definition:

Informal definition. Conformal maps are functions on **C** that preserve the angles between curves.

More precisely: Suppose $f(z)$ is differentiable at z_0 and $\gamma(t)$ is a smooth curve through z_0. To be concrete, let's suppose $\gamma(t_0) = z_0$. The function maps the point z_0 to $w_0 = f(z_0)$ and the curve γ to

$$\tilde{\gamma}(t) = f(\gamma(t)).$$

Under this map, the tangent vector $\gamma'(t_0)$ at z_0 is mapped to the tangent vector

$$\tilde{\gamma}'(t_0) = (f \circ \gamma)'(t_0)$$

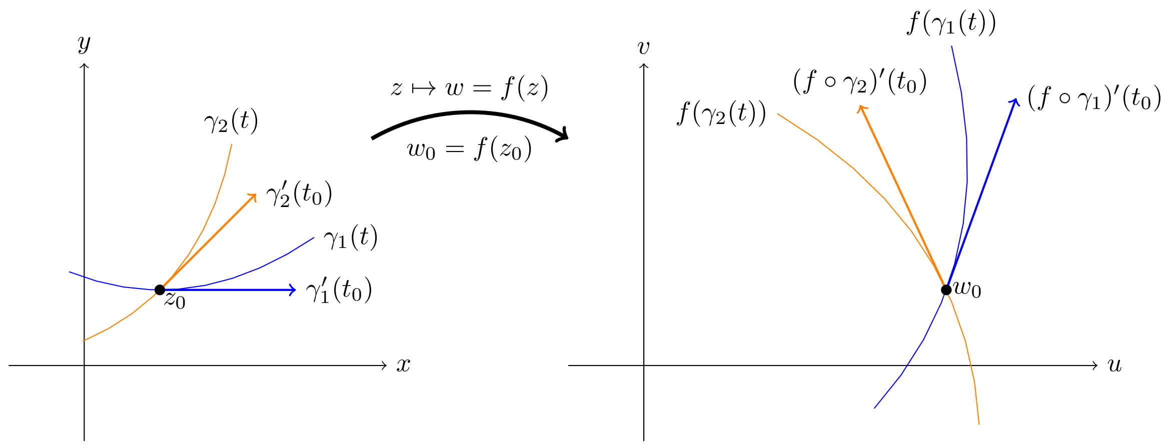
at w_0. With these notations we have the following definition.

Definition. The function $f(z)$ is conformal at z_0 if there is an angle ϕ and a scale $a > 0$ such that for any smooth curve $\gamma(t)$ through z_0 the map f rotates the tangent vector at z_0 by ϕ and scales it by a. That is, for any γ, the tangent vector $(f \circ \gamma)'(t_0)$ is found by rotating $\gamma'(t_0)$ by ϕ and scaling it by a.

If $f(z)$ is defined on a region A, we say it is a conformal map on A if it is conformal at each point z in A.

Note. The scale factor a and rotation angle ϕ depends on the point z, but not on any of the curves through z.

Example 10.1. The figure below shows a conformal map $f(z)$ mapping two curves through z_0 to two curves through $w_0 = f(z_0)$. The tangent vectors to each of the original curves are both rotated and scaled by the same amount.

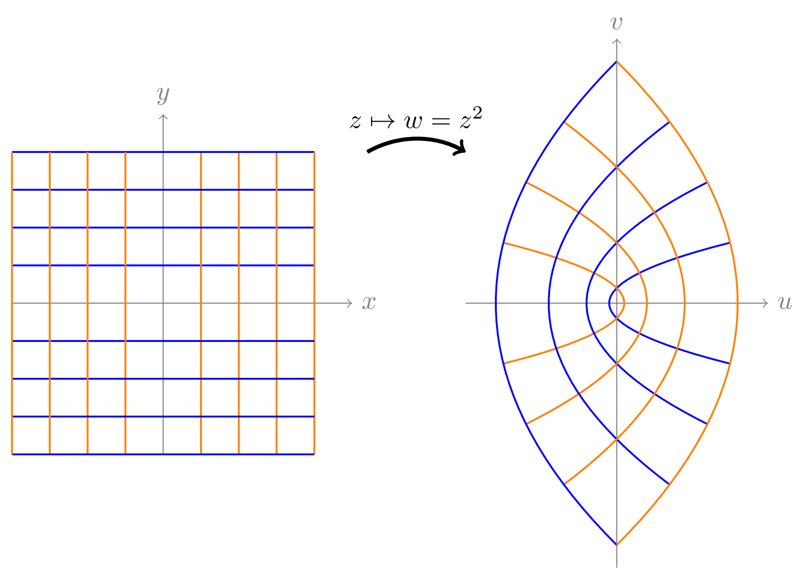


A conformal map rotates and scales all tangent vectors at z_0 by the same ammount.

Remark 1. Conformality is a local phenomenon. At a different point z_1 the rotation angle and scale factor might be different.

Remark 2. Since rotations preserve the angles between vectors, a key property of conformal maps is that they preserve the angles between curves.

Example 10.2. Recall that way back in Topic 1 we saw that $f(z) = z^2$ maps horizontal and vertical grid lines to mutually orthogonal parabolas. We will see that $f(z)$ is conformal. So, the orthogonality of the parabolas is no accident. The conformal map preserves the right angles between the grid lines.



10.3 Tangent vectors as complex numbers

In 18.02, you used parametrized curves $\gamma(t) = (x(t), y(t))$ in the xy-plane. Considered this way, the tangent vector is just the derivative:

$$\gamma'(t) = (x'(t), y'(t)).$$

Note, as a vector, (x', y') represents a *displacement*. If the vector starts at the origin, then the endpoint is at (x', y'). More typically we draw the vector starting at the point $\gamma(t)$.

In 18.04, we use parametrized curves $\gamma(t) = x(t) + iy(t)$ in the complex plane. Considered this way, the tangent vector is just the derivative:

$$\gamma'(t) = x'(t) + iy'(t).$$

It should be clear that these representations are equivalent. The vector (x', y') and the complex number $x' + iy'$ both represent the same displacement. Also, the length of a vector and the angle between two vectors is the same in both representations.

Thinking of tangent vectors to curves as complex numbers allows us to recast conformality in terms of complex numbers.

Theorem 10.3. If $f(z)$ is conformal at z_0 then there is a complex number $c = ae^{i\phi}$ such that the map f multiplies tangent vectors at z_0 by c. Conversely, if the map f multiplies all tangent vectors at z_0 by $c = ae^{i\phi}$ then f is conformal at z_0.

Proof. By definition f is conformal at z_0 means that there is an angle ϕ and a scalar $a > 0$ such that the map f rotates tangent vectors at z_0 by ϕ and scales them by a. This is exactly the effect of multiplication by $c = ae^{i\phi}$.

10.4 Analytic functions are conformal

Theorem 10.4. (Operational definition of conformal) If f is analytic on the region A and $f'(z_0) \neq 0$, then f is conformal at z_0. Furthermore, the map f multiplies tangent vectors at z_0 by $f'(z_0)$.

Proof. The proof is a quick computation. Suppose $z = \gamma(t)$ is curve through z_0 with $\gamma(t_0) = z_0$. The curve $\gamma(t)$ is transformed by f to the curve $w = f(\gamma(t))$. By the chain rule we have

$$\left.\frac{d f(\gamma(t))}{dt}\right|_{t_0} = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0).$$

The theorem now follows from Theorem 10.3.

Example 10.5. (Basic example) Suppose $c = ae^{i\phi}$ and consider the map $f(z) = cz$. Geometrically, this map rotates every point by ϕ and scales it by a. Therefore, it must have the same effect on all tangent vectors to curves. Indeed, f is analytic and $f'(z) = c$ is constant.

Example 10.6. Let $f(z) = z^2$. So $f'(z) = 2z$. Thus the map f has a different affect on tangent vectors at different points z_1 and z_2.

Example 10.7. (Linear approximation) Suppose $f(z)$ is analytic at $z = 0$. The linear approximation (first two terms of the Taylor series) is

$$f(z) \approx f(0) + f'(0)z.$$

If $\gamma(t)$ is a curve with $\gamma(t_0) = 0$ then, near t_0,

$$f(\gamma(t)) \approx f(0) + f'(0)\gamma(t).$$

That is, near 0, f looks like our basic example plus a shift by $f(0)$.

Example 10.8. The map $f(z) = \overline{z}$ has lots of nice geometric properties, but it is not conformal. It preserves the length of tangent vectors and the angle between tangent vectors. The reason it isn't conformal is that is does not rotate tangent vectors. Instead, it reflects them across the x-axis.

In other words, it reverses the orientation of a pair of vectors. Our definition of conformal maps requires that it preserves orientation.

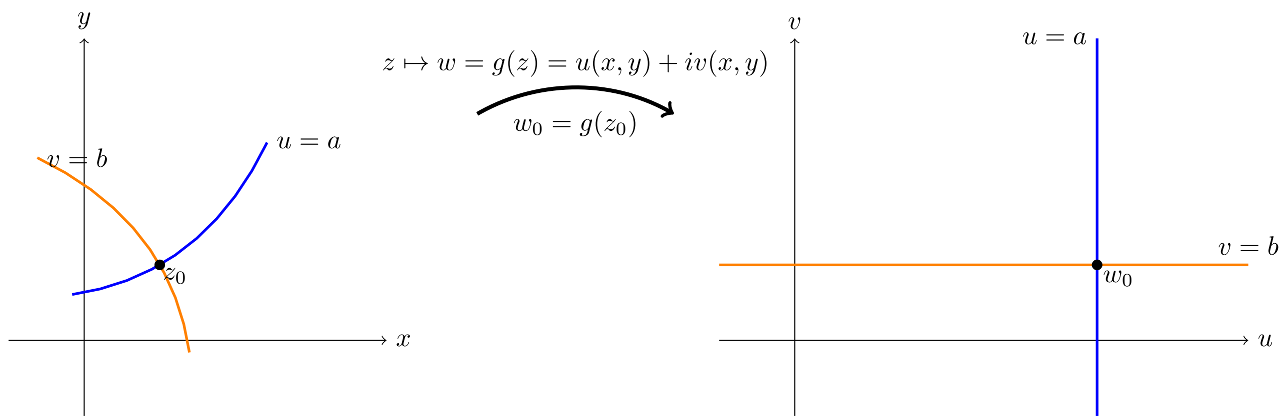
10.5 Digression to harmonic functions

Theorem 10.9. If u and v are harmonic conjugates and $g = u + iv$ has $g'(z_0) \neq 0$, then the level curves of u and v through z_0 are orthogonal.

Note. We proved this in an earlier topic using the Cauchy-Riemann equations. Here will make an argument involving conformal maps.

Proof. First we'll examine how g maps the level curve $u(x, y) = a$. Since $g = u + iv$, the image of the level curve is $w = a + iv$, i.e it's (contained in) a vertical line in the w-plane. Likewise, the level curve $v(x, y) = b$ is mapped to the horizontal line $w = u + ib$.

Thus, the images of the two level curves are orthogonal. Since g is conformal it preserves the angle between the level curves, so they must be orthogonal.



$g = u + iv$ maps level curves of u and v to grid lines.

10.6 Riemann mapping theorem

The Riemann mapping theorem is a major theorem on conformal maps. The proof is fairly technical and we will skip it. In practice, we will write down explicit conformal maps between regions.

Theorem 10.10. (Riemann mapping theorem) If A is simply connected and not the whole plane, then there is a bijective conformal map from A to the unit disk.

Corollary. For any two such regions there is a bijective conformal map from one to the other. We say they are conformally equivalent.

10.7 Fractional linear transformations

Definition. A fractional linear transformation is a function of the form

$$T(z) = \frac{az + b}{cz + d}, \quad \text{where } a,\ b,\ c,\ d \text{ are complex constants and } ad - bc \neq 0$$

These are also called Mobius transforms or bilinear transforms. We will abbreviate fractional linear transformation as FLT.

Simple point. If $ad - bc = 0$ then $T(z)$ is a constant function.

Proof. The full proof requires that we deal with all the cases where some of the coefficients are 0. We'll give the proof assuming $c \neq 0$ and leave the case $c = 0$ to you. Assuming $c \neq 0$, the condition $ad - bc = 0$ implies

$$\frac{a}{c}(c, d) = (a, b).$$

So,

$$T(z) = \frac{(a/c)(cz + d)}{cz + d} = \frac{a}{c}.$$

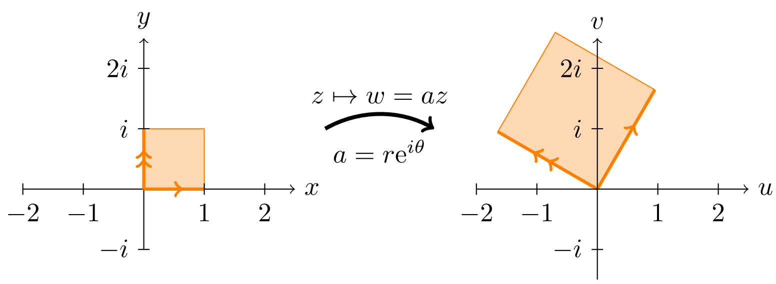
That is, $T(z)$ is constant.

Extension to ∞. It will be convenient to consider linear transformations to be defined on the extended complex plane $\mathbf{C} \cup \{\infty\}$ by defining

$$T(\infty) = \begin{cases} a/c & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases}$$
$$T(-d/c) = \infty \quad \text{if } c \neq 0.$$

10.7.1 Examples

Example 10.11. (Scale and rotate) Let $T(z) = az$. If $a = r$ is real this scales the plane. If $a = e^{i\theta}$ it rotates the plane. If $a = re^{i\theta}$ it does both at once.



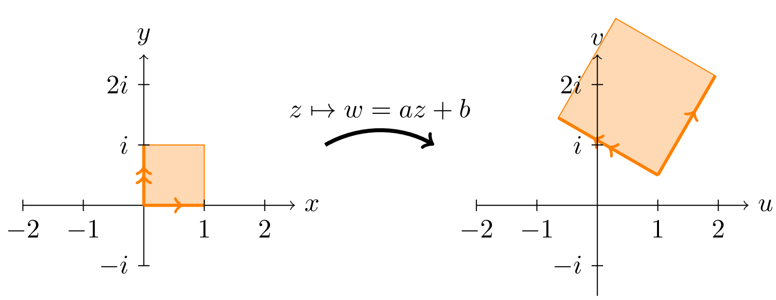
Multiplication by $a = re^{i\theta}$ scales by r and rotates by θ

Note that T is the fractional linear transformation with coefficients

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.$$

(We'll see below the benefit of presenting the coefficients in matrix form!)

Example 10.12. (Scale and rotate and translate) Let $T(z) = az + b$. Adding the b term introduces a translation to the previous example.

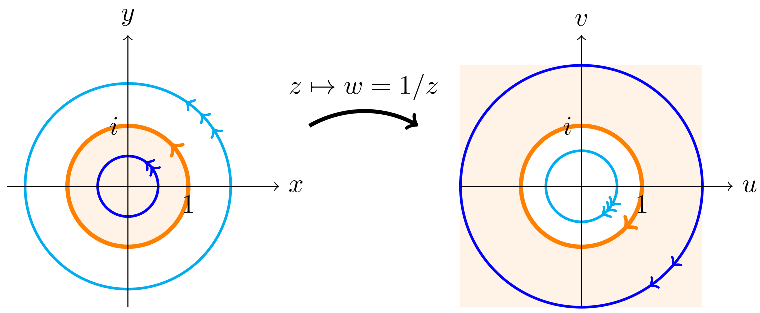


The map $w = az + b$ scales, rotates and shifts the square.

Note that T is the fractional linear transformation with coefficients

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.$$

Example 10.13. (Inversion) Let $T(z) = 1/z$. This is called an inversion. It turns the unit circle inside out. Note that $T(0) = \infty$ and $T(\infty) = 0$. In the figure below the circle that is outside the unit circle in the z plane is inside the unit circle in the w plane and vice-versa. Note that the arrows on the curves are reversed.



The map $w = 1/z$ inverts the plane.

Note that T is the fractional linear transformation with coefficients

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 10.14. Let

$$T(z) = \frac{z - i}{z + i}.$$

We claim that this maps the x-axis to the unit circle and the upper half-plane to the unit disk.

Proof. First take x real, then

$$|T(x)| = \frac{|x - i|}{|x + i|} = \frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} = 1.$$

So, T maps the x-axis to the unit circle.

Next take $z = x + iy$ with $y > 0$, i.e. z in the upper half-plane. Clearly

$$|y + 1| > |y - 1|,$$

so

$$|z + i| = |x + i(y + 1)| > |x + i(y - 1)| = |z - i|,$$

implying that

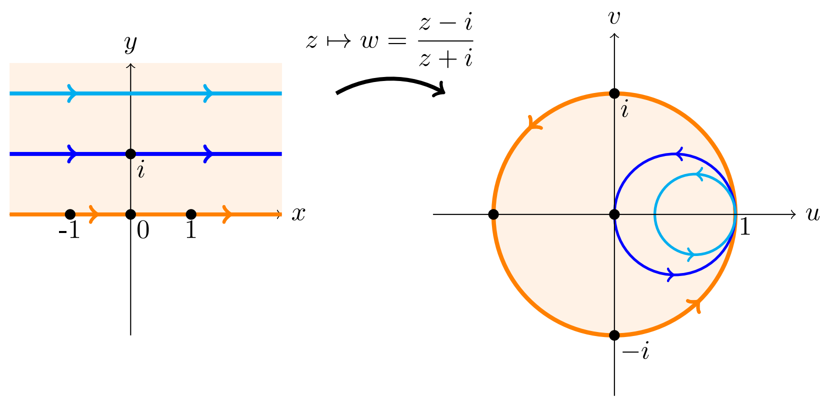
$$|T(z)| = \frac{|z - i|}{|z + i|} < 1.$$

So, T maps the upper half-plane to the unit disk.

We will use this map frequently, so for the record we note that

$$T(i) = 0, \quad T(\infty) = 1, \quad T(-1) = i, \quad T(0) = -1, \quad T(1) = -i.$$

These computations show that the real axis is mapped counterclockwise around the unit circle starting at 1 and coming back to 1.



The map $w = \dfrac{z-i}{z+i}$ maps the upper-half plane to the unit disk.

10.7.2 Lines and circles

Theorem. A linear fractional transformation maps lines and circles to lines and circles.

Before proving this, note that it does not say lines are mapped to lines and circles to circles. For example, in Example 10.14 the real axis is mapped the unit circle. You can also check that inversion $w = 1/z$ maps the line $z = 1 + iy$ to the circle $|z - 1/2| = 1/2$.

Proof. We start by showing that inversion maps lines and circles to lines and circles. Given z and $w = 1/z$ we define x, y, u and v by

$$z = x + iy \quad \text{and} \quad w = \frac{1}{z} = \frac{x - iy}{x^2 + y^2} = u + iv$$

So,

$$u = \frac{x}{x^2 + y^2} \qquad \text{and} \qquad v = -\frac{y}{x^2 + y^2}.$$

Now, every circle or line can be described by the equation

$$Ax + By + C(x^2 + y^2) = D$$

(If $C = 0$ it descibes a line, otherwise a circle.) We convert this to an equation in u, v as follows.

$$\begin{aligned}
& Ax + By + C(x^2 + y^2) = D \\
\Leftrightarrow \quad & \frac{Ax}{x^2 + y^2} + \frac{By}{x^2 + y^2} + C = \frac{D}{x^2 + y^2} \\
\Leftrightarrow \quad & Au - Bv + C = D(u^2 + v^2).
\end{aligned}$$

In the last step we used the fact that

$$u^2 + v^2 = |w|^2 = 1/|z|^2 = 1/(x^2 + y^2).$$

We have shown that a line or circle in x, y is transformed to a line or circle in u, v. This shows that inversion maps lines and circles to lines and circles.

We note that for the inversion $w = 1/z$.

1. Any line not through the origin is mapped to a circle through the origin.
2. Any line through the origin is mapped to a line through the origin.
3. Any circle not through the origin is mapped to a circle not through the origin.
4. Any circle through the origin is mapped to a line not through the origin.

Now, to prove that an arbitrary fractional linear transformation maps lines and circles to lines and circles, we factor it into a sequence of simpler transformations.

First suppose that $c = 0$. So,

$$T(z) = (az + b)/d.$$

Since this is just translation, scaling and rotating, it is clear it maps circles to circles and lines to lines.

Now suppose that $c \neq 0$. Then,

$$T(z) = \frac{az + b}{cz + d} = \frac{\frac{a}{c}(cz + d) + b - \frac{ad}{c}}{cz + d} = \frac{a}{c} + \frac{b - ad/c}{cz + d}$$

So, $w = T(z)$ can be computed as a composition of transforms

$$z \quad \mapsto \quad w_1 = cz + d \quad \mapsto \quad w_2 = 1/w_1 \quad \mapsto \quad w = \frac{a}{c} + (b - ad/c)w_2$$

We know that each of the transforms in this sequence maps lines and circles to lines and circles. Therefore the entire sequence does also. $\square$

10.7.3 Mapping z_j to w_j

It turns out that for two sets of three points z_1, z_2, z_3 and w_1, w_2, w_3 there is a fractional linear transformation that takes z_j to w_j. We can construct this map as follows.

Let

$$T_1(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

Notice that

$$T_1(z_1) = 0, \qquad T_1(z_2) = 1, \qquad T_1(z_3) = \infty.$$

Likewise let

$$T_2(w) = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}.$$

Notice that

$$T_2(w_1) = 0, \qquad T_2(w_2) = 1, \qquad T_2(w_3) = \infty.$$

Now $T(z) = T_2^{-1} \circ T_1(z)$ is the required map.

10.7.4 Correspondence with matrices

We can identify the transformation

$$T(z) = \frac{az+b}{cz+d}$$

with the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

This identification is useful because of the following algebraic facts.

1. If $r \neq 0$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $r \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ correspond to the same FLT.

Proof. This follows from the obvious equality

$$\frac{az+b}{cz+d} = \frac{raz+rb}{rcz+rd}.$$

2. If $T(z)$ corresponds to $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $S(z)$ corresponds to $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ then composition $T \circ S(z)$ corresponds to matrix multiplication AB.

Proof. The proof is just a bit of algebra.

$$T \circ S(z) = T\left(\frac{ez+f}{gz+h}\right) = \frac{a\left((ez+f)/(gz+h)\right)+b}{c\left((ez+f)/(gz+h)\right)+d} = \frac{(ae+bg)z+af+bh}{(ce+dg)z+cf+dh}$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

The claimed correspondence is clear from the last entries in the two lines above.

3. If $T(z)$ corresponds to $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then T has an inverse and $T^{-1}(w)$ corresponds to A^{-1} and also to $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, i.e. to A^{-1} without the factor of $1/\det(A)$.

Proof. Since $AA^{-1} = I$ it is clear from the previous fact that T^{-1} corresponds to A^{-1}. Since

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Fact 1 implies A^{-1} and $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ both correspond to the same FLT, i.e. to T^{-1}.

Example 10.15.

1. The matrix $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ corresponds to $T(z) = az + b$.
2. The matrix $\begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}$ corresponds to rotation by 2α.
3. The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ corresponds to the inversion $w = 1/z$.

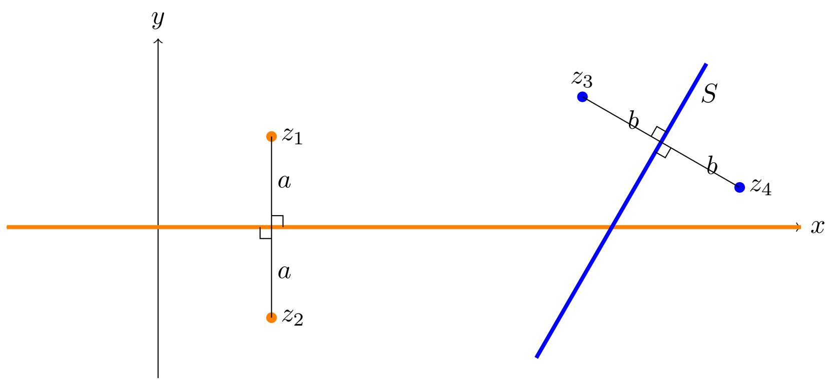
10.8 Reflection and symmetry

10.8.1 Reflection and symmetry in a line

Example 10.16. Suppose we have a line S and a point z_1 not on S. The reflection of z_1 in S is the point z_2 so that S is the perpendicular bisector to the line segment $\overline{z_1 z_2}$. Since there is exactly one such point z_2, the reflection of a point in a line is unique.

Definition. If z_2 is the reflection of z_1 in S, we say that z_1 and z_2 are symmetric with respect to the line S.

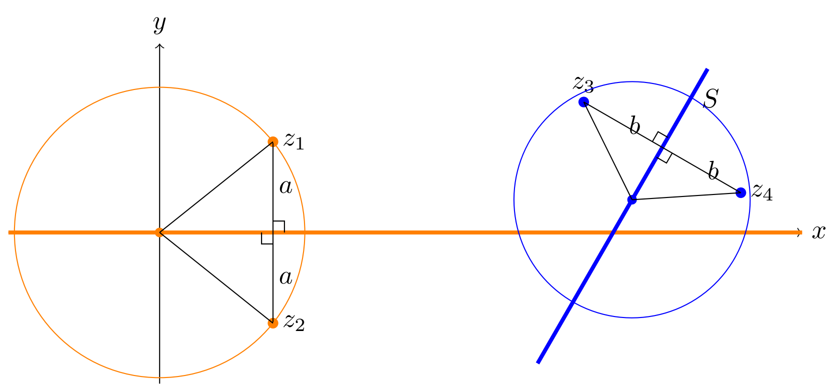
In the figure below the points z_1 and z_2 are symmetric in the x-axis. The points z_3 and z_4 are symmetric in the line S.



In order to define the reflection of a point in a circle we need to work a little harder. Looking back at the previous example we can show the following.

Fact. If z_1 and z_2 are symmetric in the line S, then any circle through z_1 and z_2 intersects S orthogonally.

Proof. Call the circle C. Since S is the perpendicular bisector of a chord of C, the center of C lies on S. Therefore S is a radial line, i.e. it intersects C orthogonally.



Circles through symmetric points intersect the line at right angles.

10.8.2 Reflection and symmetry in a circle

We will adapt this for our definition of reflection in a circle. So that the logic flows correctly we need to start with the definition of symmetric pairs of points.

Definition. Suppose S is a line or circle. A pair of points z_1, z_2 is called symmetric with respect to S if every line or circle through the two points intersects S orthogonally.

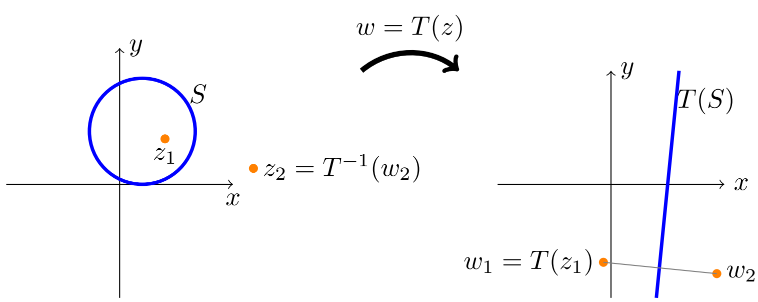
First we state an almost trivial fact.

Fact. Fractional linear transformations preserve symmetry. That is, if z_1 and z_2 are symmetric in a line or circle S, then, for an FLT T, $T(z_1)$ and $T(z_2)$ are symmetric in $T(S)$.

Proof. The definition of symmetry is in terms of lines and circles, and angles. Fractional linear transformations map lines and circles to lines and circles and, being conformal, preserve angles. □

Theorem. Suppose S is a line or circle and z_1 a point not on S. There is a unique point z_2 such that the pair z_1, z_2 is symmetric in S.

Proof. Let T be a fractional linear transformation that maps S to a line. We know that $w_1 = T(z_1)$ has a unique reflection w_2 in this line. Since T^{-1} preserves symmetry, z_1 and $z_2 = T^{-1}(w_2)$ are symmetric in S. Since w_2 is the unique point symmetric to w_1 the same is true for z_2 vis-a-vis z_1. This is all shown in the figure below.



We can now define reflection in a circle.

Definition. The point z_2 in the theorem is called the reflection of z_1 in S.

10.8.3 Reflection in the unit circle

Using the symmetry preserving feature of fractional linear transformations, we start with a line and transform to the circle. Let R be the real axis and C the unit circle. We know the FLT

$$T(z) = \frac{z-i}{z+i}$$

maps R to C. We also know that the points z and $\overline{z}$ are symmetric in R. Therefore

$$w_1 = T(z) = \frac{z-i}{z+i} \qquad \text{and } w_2 = T(\overline{z}) = \frac{\overline{z}-i}{\overline{z}+i}$$

are symmetric in D. Looking at the formulas, it is clear that $w_2 = 1/\overline{w_1}$. This is important enough that we highlight it as a theorem.

Theorem. (Reflection in the unit circle) The reflection of $z = x + iy = re^{i\theta}$ in the unit circle is

$$\frac{1}{\overline{z}} = \frac{z}{|z|^2} = \frac{x+iy}{x^2+y^2} = \frac{e^{i\theta}}{r}.$$

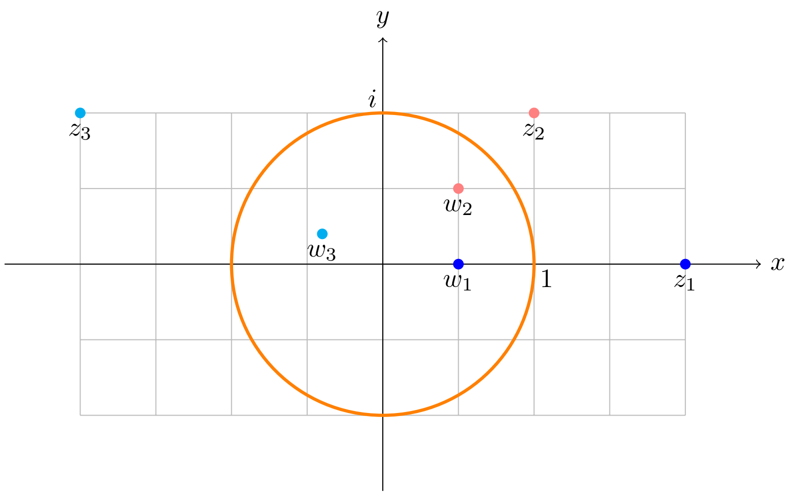
The calculations from $1/\overline{z}$ are all trivial.

Notes.

1. It is possible, but more tedious and less insightful, to arrive at this theorem by direct calculation.
2. If z is on the unit circle then $1/\overline{z} = z$. That is, z is its own reflection in the unit circle –as it should be.
3. The center of the circle 0 is symmetric to the point at ∞.

The figure below shows three pairs of points symmetric in the unit circle:

$$z_1 = 2;\ w_1 = \frac{1}{2}, \quad z_2 = 1 + i;\ w_2 = \frac{1+i}{2}, \quad z_3 = -2 + i;\ w_3 = \frac{-2+i}{5}.$$



Pairs of points z_j; w_j symmetric in the unit circle.

Example 10.17. Reflection in the circle of radius R. Suppose S is the circle $|z| = R$ and z_1 is a point not on S. Find the reflection of z_1 in S.

Solution: Our strategy is to map S to the unit circle, find the reflection and then map the unit circle back to S.

Start with the map $T(z) = w = z/R$. Clearly T maps S to the unit circle and

$$w_1 = T(z_1) = z_1/R.$$

The reflection of w_1 is

$$w_2 = 1/\overline{w_1} = R/\overline{z}_1.$$

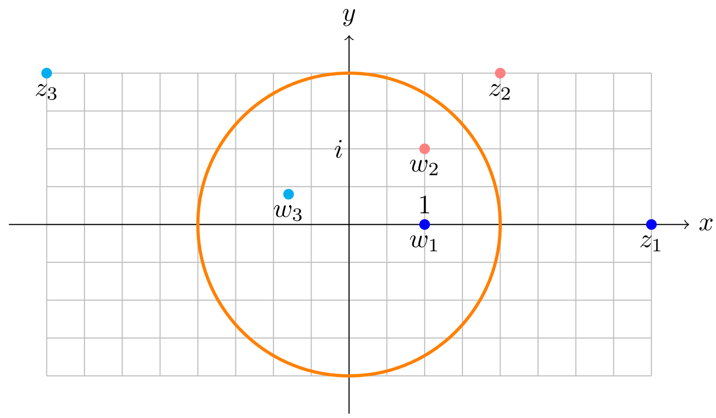
Mapping back from the unit circle by T^{-1} we have

$$z_2 = T^{-1}(w_2) = Rw_2 = R^2/\overline{z}_1.$$

Therefore the reflection of z_1 is $R^2/\overline{z}_1$.

Here are three pairs of points symmetric in the circle of radius 2. Note, that this is the same figure as the one above with everything doubled.

$$z_1 = 4;\ w_1 = 1, \qquad z_2 = 2+2i;\ w_2 = 1+i, \qquad z_3 = -4+2i;\ w_3 = \frac{-4+2i}{5}.$$



Pairs of points z_j; w_j symmetric in the circle of radius 2.

Example 10.18. Find the reflection of z_1 in the circle of radius R centered at c.

Solution: Let $T(z) = (z-c)/R$. T maps the circle centered at c to the unit circle. The inverse map is

$$T^{-1}(w) = Rw + c.$$

So, the reflection of z_1 is given by mapping z to $T(z)$, reflecting this in the unit circle, and mapping back to the original geometry with T^{-1}. That is, the reflection z_2 is

$$z_1 \to \frac{z_1 - c}{R} \to \frac{R}{\overline{z_1 - c}} \to \boxed{z_2 = \frac{R^2}{\overline{z_1 - c}} + c.}$$

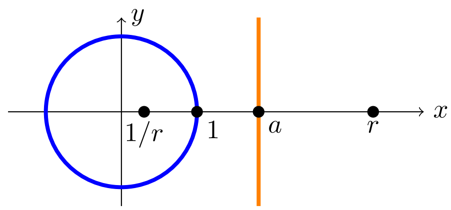
We can now record the following important fact.

Fact. (Reflection of the center) For a circle S with center c the pair c, ∞ is symmetric with respect to the circle.

Proof. This is an immediate consequence of the formula for the reflection of a point in a circle. For example, the reflection of z in the unit circle is $1/\overline{z}$. So, the reflection of 0 is infinity.

Example 10.19. Show that if a circle and a line don't intersect then there is a pair of points z_1, z_2 that is symmetric with respect to both the line and circle.

Solution: By shifting, scaling and rotating we can find a fractional linear transformation T that maps the circle and line to the following configuration: The circle is mapped to the unit circle and the line to the vertical line $x = a > 1$.



For any real r, $w_1 = r$ and $w_2 = 1/r$ are symmetric in the unit circle. We can choose a specific r so that r and $1/r$ are equidistant from a, i.e. also symmetric in the line $x = a$. It is clear geometrically that this can be done. Algebraically we solve the equation

$$\frac{r + 1/r}{2} = a \quad \Rightarrow \quad r^2 - 2ar + 1 = 0 \quad \Rightarrow \quad r = a + \sqrt{a^2 - 1} \quad \Rightarrow \quad \frac{1}{r} = a - \sqrt{a^2 - 1}.$$

Thus $z_1 = T^{-1}(a + \sqrt{a^2 - 1})$ and $z_2 = T^{-1}(a - \sqrt{a^2 - 1})$ are the required points.

Example 10.20. Show that if two circles don't intersect then there is a pair of points z_1, z_2 that is symmetric with respect to both circles.

Solution: Using a fractional linear transformation that maps one of the circles to a line (and the other to a circle) we can reduce the problem to that in the previous example.

Example 10.21. Show that any two circles that don't intersect can be mapped conformally to concentric circles.

Solution: Call the circles S_1 and S_2. Using the previous example start with a pair of points z_1, z_2 which are symmetric in both circles. Next, pick a fractional linear transformation T that maps z_1 to 0 and z_2 to infinity. For example,

$$T(z) = \frac{z - z_1}{z - z_2}.$$

Since T preserves symmetry 0 and ∞ are symmetric in the circle $T(S_1)$. This implies that 0 is the center of $T(S_1)$. Likewise 0 is the center of $T(S_2)$. Thus, $T(S_1)$ and $T(S_2)$ are concentric.

10.9 Solving the Dirichlet problem for harmonic functions

In general, a Dirichlet problem in a region A asks you to solve a partial differential equation in A where the values of the solution on the boundary of A are specificed.

Example 10.22. Find a function u harmonic on the unit disk such that

$$u(\mathrm{e}^{i\theta}) = \begin{cases} 1 & \text{for } 0 < \theta < \pi \\ 0 & \text{for } -\pi < \theta < 0 \end{cases}$$

This is a Dirichlet problem because the values of u on the boundary are specified. The partial differential equation is implied by requiring that u be harmonic, i.e. we require $\nabla^2 u = 0$. We will solve this problem in due course.

10.9.1 Harmonic functions on the upper half-plane

Our strategy will be to solve the Dirichlet problem for harmonic functions on the upper half-plane and then transfer these solutions to other domains.

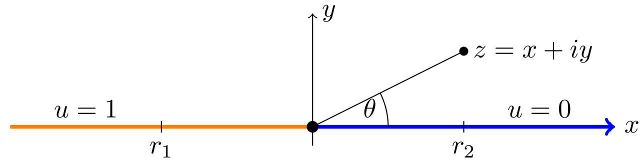
Example 10.23. Find a harmonic function $u(x, y)$ on the upper half-plane that satisfies the boundary condition

$$u(x, 0) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases}$$

Solution: We can write down a solution explicitly as

$$u(x, y) = \frac{1}{\pi}\theta,$$

where θ is the argument of $z = x + iy$. Since we are only working on the upper half-plane we can take any convenient branch with branch cut in the lower half-plane, say $-\pi/2 < \theta < 3\pi/2$.



To show u is truly a solution, we have to verify two things:

1. u satisfies the boundary conditions
2. u is harmonic.

Both of these are straightforward. First, look at the point r_2 on the positive x-axis. This has argument $\theta = 0$, so $u(r_2, 0) = 0$. Likewise $\arg(r_1) = \pi$, so $u(r_1, 0) = 1$. Thus, we have shown point (1).

To see point (2) remember that

$$\log(z) = \log(r) + i\theta.$$

So,

$$u = \text{Re}\left(\frac{1}{\pi i}\log(z)\right).$$

Since it is the real part of an analytic function, u is harmonic.

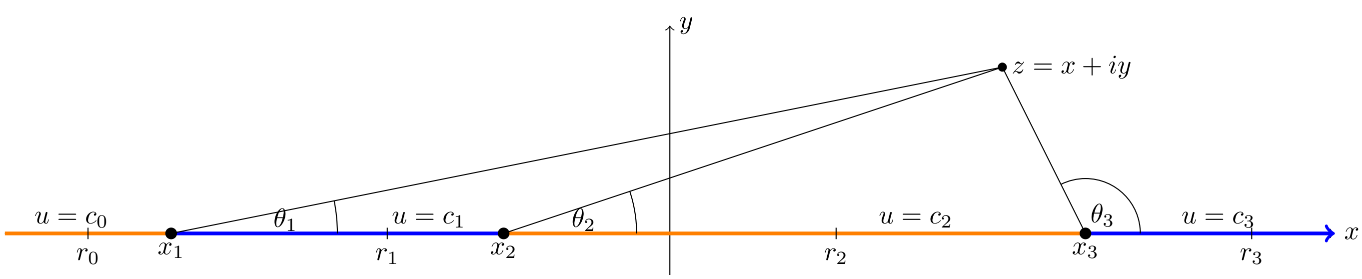
Example 10.24. Suppose $x_1 < x_2 < x_3$. Find a harmonic function u on the upper half-plane that satisfies the boundary condition

$$u(x, 0) = \begin{cases} c_0 & \text{for } x < x_1 \\ c_1 & \text{for } x_1 < x < x_2 \\ c_2 & \text{for } x_2 < x < x_3 \\ c_3 & \text{for } x_3 < x \end{cases}$$

Solution: We mimic the previous example and write down the solution

$$u(x, y) = c_3 + (c_2 - c_3)\frac{\theta_3}{\pi} + (c_1 - c_2)\frac{\theta_2}{\pi} + (c_0 - c_1)\frac{\theta_1}{\pi}.$$

Here, the θ_j are the angles shown in the figure. One again, we chose a branch of θ that has $0 < \theta < \pi$ for points in the upper half-plane. (For example the branch $-\pi/2 < \theta < 3\pi/2$.)



To convince yourself that u satisfies the boundary condition test a few points:

- At r_3: all the $\theta_j = 0$. So, $u(r_3, 0) = c_3$ as required.
- At r_2: $\theta_1 = \theta_2 = 0$, $\theta_3 = \pi$. So, $u(r_2, 0) = c_3 + c_2 - c_3 = c2$ as required.
- Likewise, at r_1 and r_0, u have the correct values.

As before, u is harmonic because it is the real part of the analytic function

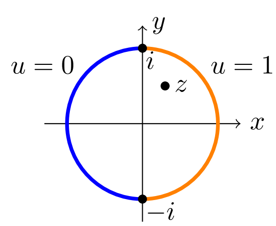
$$\Phi(z) = c_3 + \frac{(c_2 - c_3)}{\pi i}\log(z - x_3) + \frac{(c_1 - c_2)}{\pi i}\log(z - x_2) + \frac{(c_1 - c_0)}{\pi i}\log(z - x_1).$$

10.9.2 Harmonic functions on the unit disk

Let's try to solve a problem similar to the one in Example 10.22.

Example 10.25. Find a function u harmonic on the unit disk such that

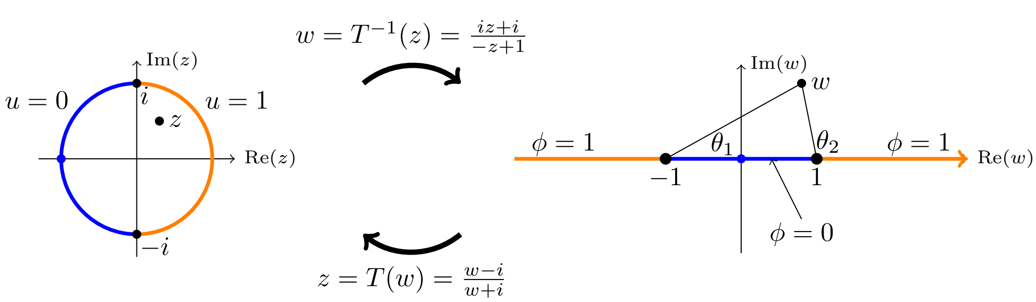
$$u(e^{i\theta}) = \begin{cases} 1 & \text{for } -\pi/2 < \theta < \pi/2 \\ 0 & \text{for } \pi/2 < \theta < 3\pi/2 \end{cases}$$



Solution: Our strategy is to start with a conformal map T from the upper half-plane to the unit disk. We can use this map to pull the problem back to the upper half-plane. We solve it there and then push the solution back to the disk.

Let's call the disk D, the upper half-plane H. Let z be the variable on D and w the variable on H. Back in Example 10.14 we found a map from H to D. The map and its inverse are

$$z = T(w) = \frac{w - i}{w + i}, \qquad w = T^{-1}(z) = \frac{iz + i}{-z + 1}.$$



The function u on D is transformed by T to a function ϕ on H. The relationships are

$$u(z) = \phi \circ T^{-1}(z) \qquad \text{or} \qquad \phi(w) = u \circ T(w)$$

These relationships determine the boundary values of ϕ from those we were given for u. We compute:

$$T^{-1}(i) = -1, \qquad T^{-1}(-i) = 1, \qquad T^{-1}(1) = \infty, \qquad T^{-1}(-1) = 0.$$

This shows the left hand semicircle bounding D is mapped to the segment $[-1, 1]$ on the real axis. Likewise, the right hand semicircle maps to the two half-lines shown. (Literally, to the 'segment' 1 to ∞ to -1.)

We know how to solve the problem for a harmonic function ϕ on H:

$$\phi(w) = 1 - \frac{1}{\pi}\theta_2 + \frac{1}{\pi}\theta_1 = \text{Re}\left(1 - \frac{1}{\pi i}\log(w-1) + \frac{1}{\pi i}\log(w+1)\right).$$

Transforming this back to the disk we have

$$u(z) = \phi \circ T^{-1}(z) = \text{Re}\left(1 - \frac{1}{\pi i}\log(T^{-1}(z)-1) + \frac{1}{\pi i}\log(T^{-1}(z)+1)\right).$$

If we wanted to, we could simplify this somewhat using the formula for T^{-1}.

10.10 Flows around cylinders

10.10.1 Milne-Thomson circle theorem

The Milne-Thomson theorem allows us to insert a circle into a two-dimensional flow and see how the flow adjusts. First we'll state and prove the theorem.

Theorem. (Milne-Thomson circle theorem) If $f(z)$ is a complex potential with all its singularities outside $|z| = R$ then

$$\Phi(z) = f(z) + \overline{f\left(\frac{R^2}{\overline{z}}\right)}$$

is a complex potential with streamline on $|z| = R$ and the same singularities as f in the region $|z| > R$.

Proof. First note that $R^2/\overline{z}$ is the reflection of z in the circle $|z| = R$.

Next we need to see that $\overline{f(R^2/\overline{z})}$ is analytic for $|z| > R$. By assumption $f(z)$ is analytic for $|z| \le R$, so it can be expressed as a Taylor series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots \tag{1}$$

Therefore,

$$\overline{f\left(\frac{R^2}{\overline{z}}\right)} = \overline{a_0} + \overline{a_1}\frac{R^2}{z} + \overline{a_2}\left(\frac{R^2}{z}\right)^2 + \ldots \tag{2}$$

All the singularities of f are outside $|z| = R$, so the Taylor series in Equation 1 converges for $|z| \le R$. This means the Laurent series in Equation 2 converges for $|z| \ge R$. That is, $\overline{f(R^2/\overline{z})}$ is analytic for $|z| \ge R$, i.e. it introduces no singularities to $\Phi(z)$ outside $|z| = R$.

The last thing to show is that $|z| = R$ is a streamline for $\Phi(z)$. This follows because for $z = Re^{i\theta}$

$$\Phi(Re^{i\theta}) = f(Re^{i\theta}) + \overline{f(Re^{i\theta})}$$

is real. Therefore

$$\psi(Re^{i\theta}) = \text{Im}(\Phi(Re^{i\theta}) = 0.$$

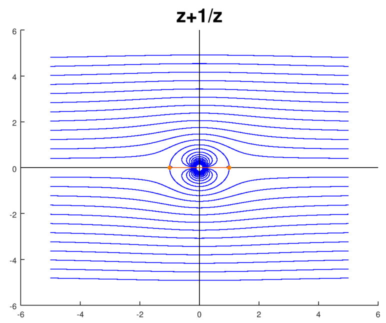
10.10.2 Examples

Think of $f(z)$ as representing flow, possibly with sources or vortices outside $|z| = R$. Then $\Phi(z)$ represents the new flow when a circular obstacle is placed in the flow. Here are a few examples.

Example 10.26. (Uniform flow around a circle) We know from Topic 6 that $f(z) = z$ is the complex potential for uniform flow to the right. So,

$$\Phi(z) = z + R^2/z$$

is the potential for uniform flow around a circle of radius R centered at the origin.



Uniform flow around a circle

Just because they look nice, the figure includes streamlines inside the circle. These don't interact with the flow outside the circle.

Note, that as z gets large flow looks uniform. We can see this analytically because

$$\Phi'(z) = 1 - R^2/z^2$$

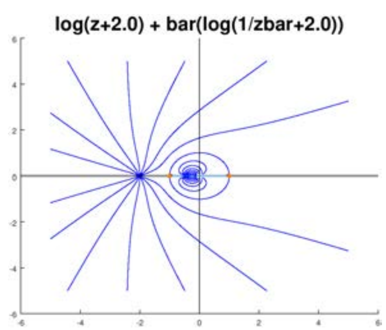
goes to 1 as z gets large. (Recall that the velocity field is (ϕ_x, ϕ_y), where $\Phi = \phi + i\psi \dots$)

Example 10.27. (Source flow around a circle) Here the source is at $z = -2$ (outside the unit circle) with complex potential

$$f(z) = \log(z + 2).$$

With the appropriate branch cut the singularities of f are also outside $|z| = 1$. So we can apply Milne-Thomson and obtain

$$\Phi(z) = \log(z + 2) + \overline{\log\left(\frac{1}{\overline{z}} + 2\right)}$$



Source flow around a circle

We know that far from the origin the flow should look the same as a flow with just a source at $z = -2$. Let's see this analytically. First we state a useful fact:

Useful fact. If $g(z)$ is analytic then so is $h(z) = \overline{g(\overline{z})}$ and $h'(z) = \overline{g'(\overline{z})}$.

Proof. Use the Taylor series for g to get the Taylor series for h and then compare $h'(z)$ and $\overline{g'(\overline{z})}$. □

Using this we have

$$\Phi'(z) = \frac{1}{z+2} - \frac{1}{z(1+2z)}$$

For large z the second term decays much faster than the first, so

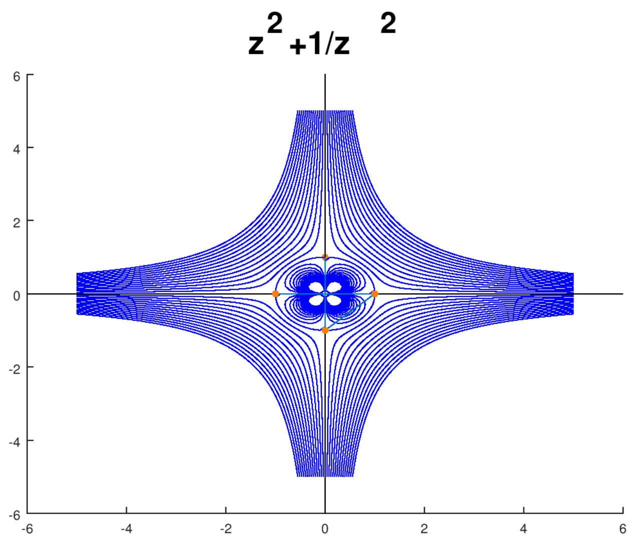
$$\Phi'(z) \approx \frac{1}{z+2}.$$

That is, far from $z = 0$, the velocity field looks just like the velocity field for $f(z)$, i.e. the velocity field of a source at $z = -2$.

Example 10.28. (Transforming flows) If we use

$$g(z) = z^2$$

we can transform a flow from the upper half-plane to the first quadrant



Source flow around a quarter circular corner

10.11 Examples of conformal maps and excercises

As we've seen, once we have flows or harmonic functions on one region, we can use conformal maps to map them to other regions. In this section we will offer a number of conformal maps between various regions. By chaining these together along with scaling, rotating and shifting we can build a large library of conformal maps. Of course there are many many others that we will not touch on.

For convenience, in this section we will let

$$T_0(z) = \frac{z-i}{z+i}.$$

This is our standard map of taking the upper half-plane to the unit disk.

Example 10.29. Let H_α be the half-plane above the line

$$y = \tan(\alpha)x,$$

i.e., $\{(x, y) \;:\; y > \tan(\alpha)x\}$. Find an FLT from H_α to the unit disk.

Solution: We do this in two steps. First use the rotation

$$T_{-\alpha}(z) = \mathrm{e}^{-i\alpha} z$$

to map H_α to the upper half-plane. Follow this with the map T_0. So our map is $T_0 \circ T_{-\alpha}(z)$.

You supply the picture

Example 10.30. Let A be the channel $0 \le y \le \pi$ in the xy-plane. Find a conformal map from A to the upper half-plane.

Solution: The map $f(z) = \mathrm{e}^z$ does the trick. (See the Topic 1 notes!)

You supply the picture: horizontal lines get mapped to rays from the origin and vertical segments in the channel get mapped to semicircles.

Example 10.31. Let B be the upper half of the unit disk. Show that T_0^{-1} maps B to the second quadrant.

Solution: You supply the argument and figure.

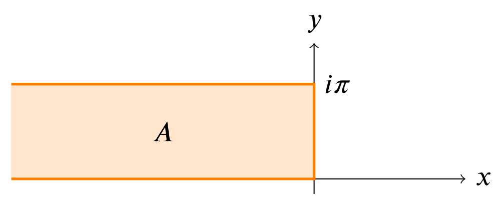
Example 10.32. Let B be the upper half of the unit disk. Find a conformal map from B to the upper half-plane.

Solution: The map $T_0^{-1}(z)$ maps B to the second quadrant. Then multiplying by $-i$ maps this to the first quadrant. Then squaring maps this to the upper half-plane. In the end we have

$$f(z) = \left(-i\left(\frac{iz + i}{-z + 1}\right)\right)^2.$$

You supply the sequence of pictures.

Example 10.33. Let A be the infinite well $\{(x, y) \;:\; x \le 0,\, 0 \le y \le \pi\}$. Find a conformal map from A to the upper half-plane.



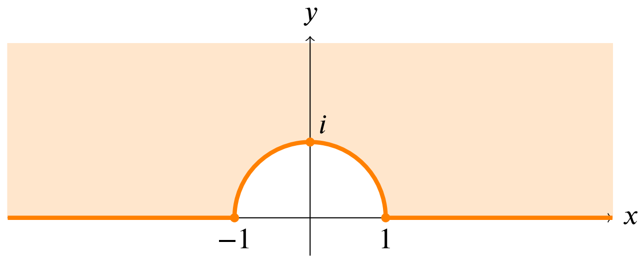
Solution: The map $f(z) = \mathrm{e}^z$ maps A to the upper half of the unit disk. Then we can use the map from Example 10.32 to map the half-disk to the upper half-plane.

You supply the sequence of pictures.

Example 10.34. Show that the function

$$f(z) = z + 1/z$$

maps the region shown below to the upper half-plane.



Solution: You supply the argument and figures