Answers to Problem Set Number 2 for 18.04. MIT (Fall 1999)

Rodolfo R. Rosales* Boris Schlittgen† Zhaohui Zhang‡

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Contents

*MIT, Department of Mathematics, room 2-337, Cambridge, MA 02139.

†MIT, Department of Mathematics, room 2-490, Cambridge, MA 02139.

‡MIT, Department of Mathematics, room 2-229, Cambridge, MA 02139.

1 Problems from the book by Saff and Snider.

1.1 Problem 04 in section 2.2.

- If $|z_0| < 1$, we have $|z_0^n - 0| = |z_0|^n \to 0$ as $n \to \infty$, hence $\lim_{n\to\infty} z_0^n = 0$, by definition (clearly, definition 1 — in page 46, section 2.2, of the book — is satisfied).

- If $|z_0| > 1$, then (given any $z \in \mathbf{C}$) we have $|z_0^n - z| > |z_0|^n - |z| \to +\infty$ as $n \to \infty$, thus z_0^n cannot converge to any $z \in \mathbf{C}$, i.e.: it diverges. Actually, many times the statement that a sequence $\{\chi_n\}$ "diverges" is used with the meaning that $|\chi_n| \to \infty$ as $n \to \infty$, which is a narrower meaning than simply not converging to any $z \in \mathbf{C}$. Clearly, this is the case here too.

1.2 Problem 04 in section 2.3.

- **a)** Assume that $f(z) = \mathrm{Re}(z)$ is differentiable at a point $z_0 \in \mathbf{C}$ and write $\Delta z = \Delta x + i\Delta y$, where $\Delta x \in \mathbf{R}$ and $\Delta y \in \mathbf{R}$. Then it should be that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x}{\Delta x + i\Delta y}\,.$$

However, if we take the path $\{\Delta x = 0\,,\ \Delta y \to 0\}$, this limit is 0. On the other hand, if we take the path $\{\Delta y = 0\,,\ \Delta x \to 0\}$, this limit is 1. This is a contradiction; thus the derivative cannot exist.

- **b)** Note that $\mathrm{Re}(z) = z - i\mathrm{Im}(z)$. Thus, if $\mathrm{Im}(z)$ is differentiable at some point, it follows that $\mathrm{Re}(z)$ is differentiable at the same point (since $f(z) = z$ is differentiable everywhere). This contradicts part **(a)**.

  Alternatively, the exact same approach used in part **(a)** can be used for this part **(b)**.

- **c)** Using that $\bar{z} = \dfrac{|z|^2}{z}$ and the fact that $g(z) = \bar{z}$ is nowhere differentiable (see example 2.3.2 in the book, also shown in the lectures), we can conclude that $f(z) = |z|$ is not differentiable at any point where $z \neq 0$ (else g would be differentiable there). At $z = 0$:

$$\frac{|z + \Delta z| - |z|}{\Delta z} = \frac{|\Delta z|}{\Delta z} = e^{-i\theta}\,, \quad \text{where} \quad \Delta z = re^{i\theta}\,.$$

Clearly, this has no limit as $\Delta z \to 0$. We can also consider directly the definition of derivative. For $f(z) = |z|$ to have a derivative at an arbitrary point $z_0 \in \mathbf{C}$, the limit (as $\Delta z \to 0$) of

$$W = \frac{|z_0 + \Delta z| - |z_0|}{|\Delta z|}$$

must exist. However, write $z_0 = re^{i\theta}$, where we can assume that $r > 0$ (we have already shown above that W has no limit when $z_0 = 0$). Then (for $0 < \rho < r$) if we take $\Delta z = \rho e^{i\theta}$, we have $W = 1$, while $\Delta z = -\rho e^{i\theta}$ yields $W = -1$. Thus, there is no limit.

1.3 Problem 16 in section 2.3.

First note that $z_1^3 = z_2^3 = 1$, so that $f(z_1) = f(z_2)$. Also $f'(z) = 3z^2$. Thus, if $f'(w) = \dfrac{f(z_2) - f(z_1)}{z_2 - z_1}$, it must be $w = 0$, which is not a point on the line segment from z_1 to z_2.

1.4 Problem 01 in section 2.4.

- **a)** For $w = f(z) = \bar{z} = x - iy$, we have $u(x, y) = x$ and $v(x, y) = -y$. Thus

  $$\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = -1\,.$$

  It follows that the 1^{st} Cauchy–Riemann condition is never satisfied, so that $w = f(z)$ is not analytic anywhere. The 2^{nd} Cauchy–Riemann condition is satisfied, but this is not enough, **both** conditions are needed for analyticity.

- **b)** For $w = f(z) = \text{Re}(z) = x$, we have $u(x, y) = x$ and $v(x, y) = 0$. Thus

  $$\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0\,.$$

  It follows that the 1^{st} Cauchy–Riemann condition is never satisfied, so that $w = f(z)$ is not analytic anywhere. The 2^{nd} Cauchy–Riemann condition is satisfied, but this is not enough, **both** conditions are needed for analyticity.

- **b)** For $w = f(z) = 2y - ix$, we have $u(x, y) = 2y$ and $v(x, y) = -x$. Thus

  $$\frac{\partial u}{\partial y} = 2 \quad \text{and} \quad \frac{\partial v}{\partial x} = -1\,.$$

  It follows that the 2^{nd} Cauchy–Riemann condition is never satisfied, so that $w = f(z)$ is not analytic anywhere. The 1^{st} Cauchy–Riemann condition is satisfied, but this is not enough, **both** conditions are needed for analyticity.

1.5 Problem 16 in section 2.4.

- **a)** We have

$$\frac{\partial x}{\partial \eta} = \frac{1}{2}, \quad \frac{\partial x}{\partial \xi} = \frac{1}{2}, \quad \frac{\partial y}{\partial \eta} = \frac{i}{2} \quad \text{and} \quad \frac{\partial y}{\partial \xi} = \frac{-i}{2}.$$

Thus, using the chain rule, we find:

$$\frac{\partial \tilde{f}}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y},$$

$$\frac{\partial \tilde{f}}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}.$$

Substituting now $f = u + iv$, the desired result follows.

- **b)** $\boxed{\frac{\partial \tilde{f}}{\partial \eta} = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)}$ — as shown in part **(a)**. Thus, it is quite clear that **the Cauchy–Riemann equations are exactly the same** as the condition $\boxed{\frac{\partial \tilde{f}}{\partial \eta} = 0}$.

1.6 Problem 02 in section 2.5.

Let $P(x, y) = ax^2 + bxy + cy^2$. For P to be harmonic, it must satisfy Laplace's equation

$$0 = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 2a + 2c.$$

Hence $\boxed{P \text{ is harmonic if and only if } a + c = 0}$ (notice that we do not have to worry about continuity of the partial derivatives, since this is trivially true for polynomials).

1.7 Problem 11 in section 2.5.

We have $f(z) = z + \frac{1}{z}$ with $z = x + iy$, where $x \in \mathbf{R}$ and $y \in \mathbf{R}$. Thus

$$\text{Im}(f(z)) = \text{Im}(z + \frac{1}{z}) = \text{Im}(x + iy + \frac{x - iy}{x^2 + y^2}) = y(1 - \frac{1}{x^2 + y^2}).$$

Thus, the level curve $\text{Im}(f(z)) = 0$ corresponds to the set of points (x, y) in the plane satisfying either

$$y = 0 \qquad \text{or} \qquad 1 - \frac{1}{x^2 + y^2} = 0.$$

That is: the union of the real axis and the unit circle.

1.8 Problem 18 in section 2.5.

Let $f(x, y) = \phi_x - i\phi_y = u + iv$. If ϕ is harmonic, then f satisfies the Cauchy–Riemann conditions because

$$u_x = \phi_{xx} = -\phi_{yy} = v_y \quad \text{and} \quad u_y = \phi_{xy} = \phi_{yx} = -v_x\,.$$

Here the first equation follows from the fact that ϕ satisfies Laplace's equation and the second is just a general property of the second partial derivatives of functions where these derivatives are continuous.

Since the partial derivatives of u and v are continuous, it follows that f is analytic.

1.9 Problem 13 in section 3.1.

For $z = x + iy$ (with x and y real), we have

$$\begin{aligned} \cos(z) &= \tfrac{1}{2}(e^{iz} + e^{-iz}) \\ &= \tfrac{1}{2}\left((e^{y} + e^{-y})\cos(x) - i(e^{y} - e^{-y})\sin(x)\right) \\ &= \cosh(y)\cos(x) - i\sinh(y)\sin(x)\,, \end{aligned}$$

where we have used the definition of the exponential function in the complex plane, which yields: $\boxed{e^{iz} = e^{-y}(\cos(x) + i\sin(x))}$ and $\boxed{e^{-iz} = e^{y}(\cos(x) - i\sin(x))}$.

The equality

$$\sin(z) = \cosh(y)\sin(x) + i\sinh(y)\cos(x)$$

follows in the same fashion.

1.10 Problem 15 in section 3.1.

Using the expression for the cosine function in terms of the exponential, we have

$$\begin{aligned} \cos(z) = 0 \quad &\Longleftrightarrow \quad e^{iz} + e^{-iz} = 0 \\ &\Longleftrightarrow \quad e^{2iz} + 1 = 0 \\ &\Longleftrightarrow \quad 2z = \pi + 2k\pi\,, \quad \text{where} \quad k \in \mathbf{Z} \\ &\Longleftrightarrow \quad z = \tfrac{1}{2}\pi + k\pi\,, \quad \text{where} \quad k \in \mathbf{Z}. \end{aligned}$$

Here we have used that $e^{\zeta} = -1$ if and only if $\zeta = i(\pi + 2k\pi)$. This follows easily from the expression for the exponential: $e^{\zeta} = e^{x}(\cos(y) + i\sin(y))$ when $\zeta = x + iy$, with x and y real.

1.11 Problem 18 in section 3.1.

- **a)** To prove this part we use the fact that $\sin(z)$ is an entire function whose derivative is $\cos(z)$. Thus, taking the derivative at $z = 0$, we get

$$\lim_{z \to 0} \frac{\sin(z)}{z} = \sin'(0) = \cos(0) = 1\,,$$

where we have used that $\sin(0) = 0$.

- **b)** For this part we use the fact that $\cos(z)$ is entire with derivative $-\sin(z)$. Thus, taking the derivative at $z = 0$, we get

$$\lim_{z \to 0} \frac{\cos(z) - 1}{z} = \cos'(0) = -\sin(0) = 0\,,$$

where we have used that $\cos(0) = 1$.

2 Other problems.

2.1 Problem 2.1 in 1999.

Statement: Consider the multiple valued mapping in the complex plane given by:

$$z \longrightarrow z^{-1/3}\,.$$

What are the images, under this map, of

1) The half plane: $\text{Re}(z) > 0$?.

2) The quadrant: $\text{Re}(z) < 0$ and $\text{Im}(z) < 0$?

3) The wedge: $-\frac{\pi}{4} < \text{Arg}(z) < \frac{\pi}{4}$?

In each case, draw the initial set and the image set and explain your answer.

Solution: When $z \neq 0$, there are three values for $z^{1/3}$, all with the same length and with their arguments $(2/3)\pi$ apart. Thus the **image of any set will consist of three parts:** if we call one of them S_0, then the other two can be obtained by rotating S_0 by $(2/3)\pi$ and $(4/3)\pi$ — note that a rotation by $(4/3)\pi$ is equivalent to one by $(-2/3)\pi$.

In the three cases here, the initial sets are all open wedges and we can obtain S_0 by "shrinking" the values of the angles by a factor of $(1/3)$. That is: if the initial set is defined by $\theta_0 < \arg(z) < \theta_1$ (where $\theta_0 < \theta_1$), then S_0 is defined by $\frac{1}{3}\theta_0 < \arg(z) < \frac{1}{3}\theta_1$. Thus we have (see the figures):

1) Image of the half plane: Re$(z) > 0$. S_0, S_1 and S_2 are defined by:

$$-\frac{1}{6}\pi < \arg(z) < \frac{1}{6}\pi\,, \qquad \frac{1}{2}\pi < \arg(z) < \frac{5}{6}\pi \qquad \text{and} \qquad -\frac{5}{6}\pi < \arg(z) < -\frac{1}{2}\pi\,,$$

respectively. See figure 2.1.1.

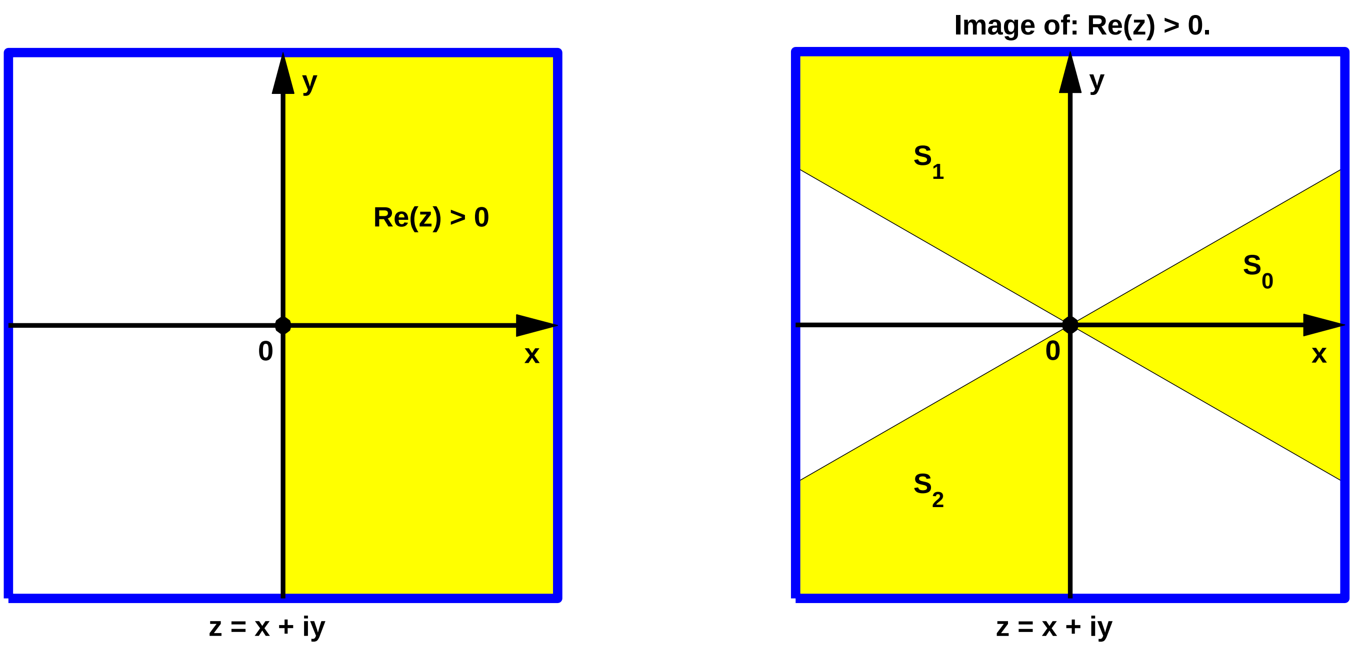

Figure 2.1.1: Image by $z^{1/3}$ of the region: Re$(z) > 0$.

2) Image of the quadrant: Re$(z) < 0$ and Im$(z) < 0$. S_0, S_1 and S_2 are defined by:

$$-\frac{1}{3}\pi < \arg(z) < -\frac{1}{6}\pi\,, \qquad \frac{1}{3}\pi < \arg(z) < \frac{1}{2}\pi \qquad \text{and} \qquad -\pi < \arg(z) < -\frac{5}{6}\pi\,,$$

respectively. See figure 2.1.2.

3) Image of the wedge: $-\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}$. S_0, S_1 and S_2 are defined by:

$$-\frac{1}{12}\pi < \arg(z) < \frac{1}{12}\pi\,, \qquad \frac{7}{12}\pi < \arg(z) < \frac{3}{4}\pi \qquad \text{and} \qquad -\frac{3}{4}\pi < \arg(z) < -\frac{7}{12}\pi\,,$$

respectively. See figure 2.1.3.

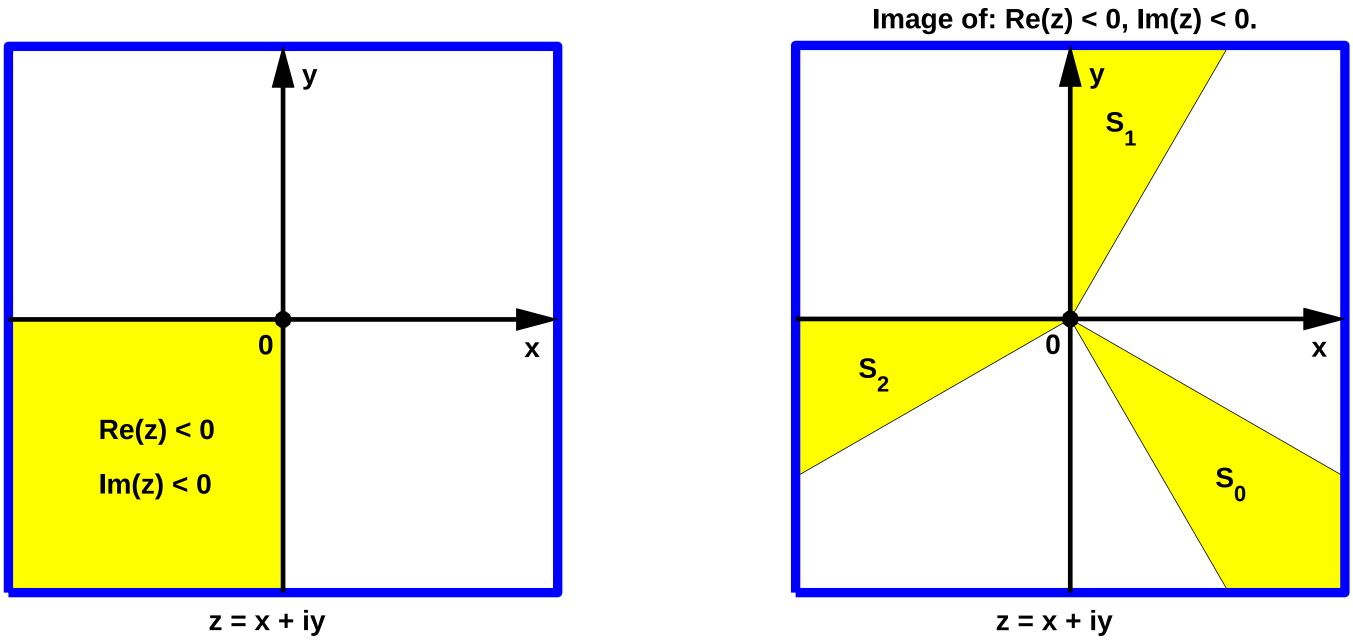

Figure 2.1.2: Image by $z^{1/3}$ of the region: Re$(z) < 0$ and Im$(z) < 0$.

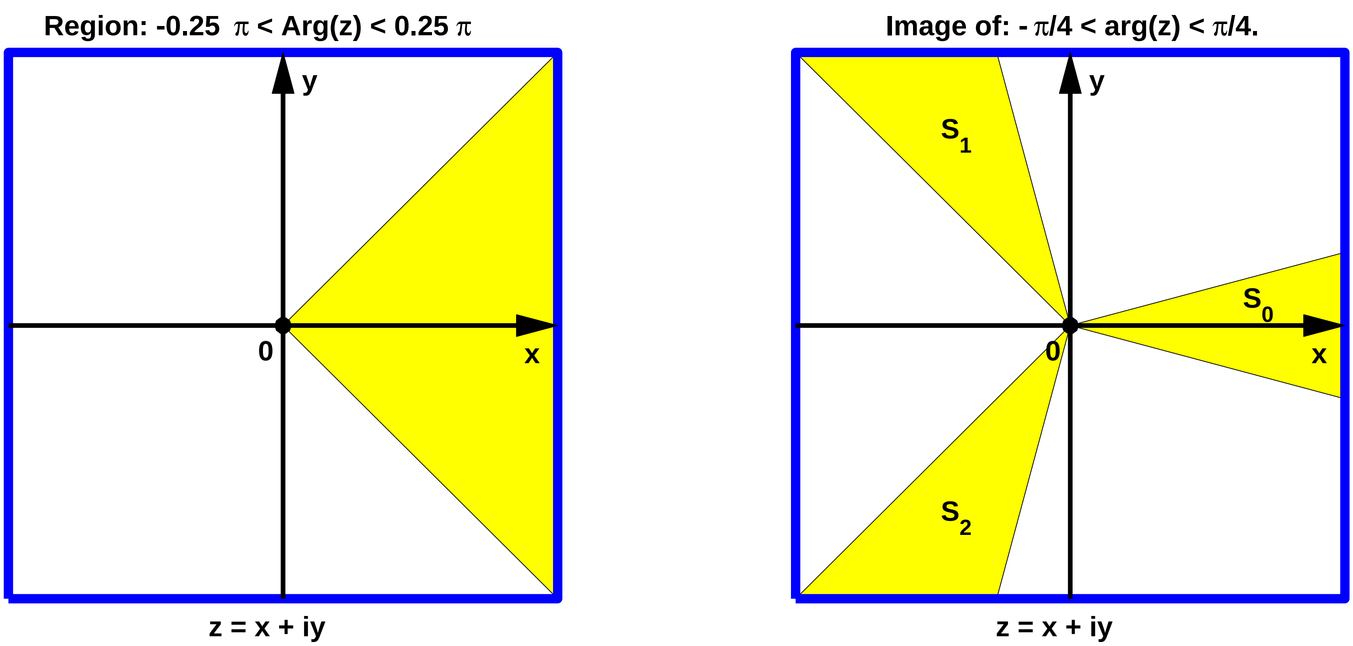

Figure 2.1.3: Image by $z^{1/3}$ of the region: $|\text{arg}(z)| < \pi/4$.

2.2 Problem 2.2 in 1999.

Statement: Consider the sequence generated by Newton's method, when computing $\sqrt{1}$. Starting from some arbitrary complex number z_0, the sequence is given by:

$$z_{n+1} = \frac{1}{2}\left(z_n + \frac{1}{z_n}\right) . \tag{2.2.1}$$

Show that if $\mathrm{Re}(z_0) > 0$, then $\mathrm{Re}(z_n) > 0$ for **all** n.

Solution: Write $z_n = x_n + iy_n$, where $x_n \in \mathbf{R}$ and $y_n \in \mathbf{R}$. Then, taking real parts in equation (2.2.1), we find:

$$x_{n+1} = \frac{1}{2}\,x_n + \frac{1}{2}\,\frac{x_n}{x_n^2 + y_n^2} = \frac{1}{2}\,x_n\left(1 + \frac{1}{x_n^2 + y_n^2}\right) .$$

Since $1 + \dfrac{1}{x_n^2 + y_n^2} > 0$ always, we have that: $x_n > 0 \implies x_{n+1} > 0$. Thus, using induction:

$$\mathrm{Re}(z_0) = x_0 > 0 \quad\implies\quad \mathrm{Re}(z_n) = x_n > 0, \quad \text{for all}\ \ n = 0, 1, 2 \ldots .$$

One little detail: we have assumed here that $x_n^2 + y_n^2 > 0$. How do we know this? Well, it is certainly true for $n = 0$, because $x_0 > 0$ (by assumption). But then $x_1 > 0$ and so $x_1^2 + y_1^2 > 0$, etc. That is: $x_n^2 + y_n^2 > 0$ is just a part of the induction argument above (none of the z_n's vanishes).

2.3 Problem 2.3 in 1999.

Statement: In problem (2.2), assume that z_0 is purely imaginary. Then **all** the z_n's are imaginary and the sequence reduces to:

$$z_n = iy_n\,, \quad \text{where} \quad y_{n+1} = \frac{1}{2}(y_n - \frac{1}{y_n}) \tag{2.3.1}$$

and the y_n's are all real. **Show that**, for $y_n = \cot(\theta_n)$, the sequence becomes $\theta_{n+1} = 2\,\theta_n$.

Now think of what happens if you take an arbitrary point on the unit circle and you move it by duplicating its argument each time. *What does this tell you about what the iterates by Newton's method do on the imaginary axis?*

Solution: Using the the formula we obtained in the answer to problem (2.2) for the real parts of the iterates, namely:

$$x_{n+1} = \frac{1}{2}\,x_n\left(1 + \frac{1}{x_n^2 + y_n^2}\right) ,$$

we see that if $x_n = 0$, then $x_{n+1} = 0$. By induction, we conclude that all the z_n are purely imaginary if z_0 is. In this case we can write

$$z_n = i\, y_n = i\, \cot(\theta_n)\,,$$

for some θ_n (where the y_n's satisfy equation (2.3.1)

Note 2.3.1 *Notice that θ_n is not uniquely defined by y_n: you can always add a multiple of 2π to a possible value and obtain another acceptable value. So, keep in mind that for each n, there is a whole bunch of possible θ_n's. We are just picking (arbitrarily) one of them.*

Substituting $y_n = \cot(\theta_n)$ into (2.3.1), we find:

$$y_{n+1} = \frac{1}{2}\left(\cot(\theta_n) - \frac{1}{\cot(\theta_n)}\right) = \frac{1}{2}\left(\frac{\cos(\theta_n)}{\sin(\theta_n)} - \frac{\sin(\theta_n)}{\cos(\theta_n)}\right) = \frac{\cos^2(\theta_n) - \sin^2(\theta_n)}{\sin(\theta_n)\cos(\theta_n)} = \cot(2\theta_n)\,.$$

It follows that $2\theta_n$ is a possible value for θ_{n+1}. So, if we $\boxed{\text{choose a } \theta_0 \text{ such that } y_0 = \cot(\theta_0),}$ then (by induction) we have

$$y_n = \cot(2^n\theta_0)\,.$$

Note 2.3.2 *Let us investigate what the sequence $\{\theta_n = 2^n\theta_0\}_{n=0}^{\infty}$ does in the set of angles. Since two angles are the same when they differ by a multiple of 2π, we are only interested in the values of this sequence modulo multiples of 2π — we write $\boxed{\theta_p = \theta_q \pmod{2\pi}}$ when θ_p and θ_p differ by a multiple of 2π.*

Let us choose θ_0 of the form $\boxed{\theta_0 = 2\mu\pi}$, where $\boxed{0 \le \mu < 1}$ (we can always do this, without loosing any generality). Then, for $n = 0, 1, 2, 3 \ldots$, **define:**

$$I_n = \textbf{Integer Part}(2^n\mu)$$

and **replace the sequence** *$\theta_n = 2^n\theta_0$ by the equivalent one given by $\phi_n = 2^n\theta_0 - 2I_n\pi$. This we can do because $\boxed{\theta_n = \phi_n \pmod{2\pi} \text{ for every } n}$. The advantage of doing this is that we have:*

$$\phi_n = 2\,(2^n\mu - I_n)\,\pi\,, \quad \textit{where} \quad 0 \le \mu_n = 2^n\mu - I_n < 1\,,\ \textit{(note that } \mu_0 = \mu\textit{)}.$$

Thus, we have "normalized" the sequence and we no longer have to worry about equivalences modulo multiples of 2π.

First **consider the case when $\mu = \dfrac{p}{q}$ is a rational number** *(here $p \geq 0$ and $q > 0$ are integers). Then*

$$\mu_n = 2^n \, \frac{p}{q} - I_n = \frac{2^n p - I_n q}{q}\,.$$

Since we also know that $0 \leq \mu_n < 1$, we can conclude that μ_n can only take values in the finite set $\{0, \frac{1}{q}, \frac{2}{q} \dots \frac{q-1}{q}\}$ — though some values may be missing. It is then not too hard to see that the sequence will have to be periodic, with some period $T < q$ (for example, if $\mu = \frac{1}{3}$, the sequence of μ_n's is given by: $\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \dots$ — the period here is: $T = 2$). Thus, **in this case the sequence of Newton iterates in (2.3.1) will wonder periodically over a finite set of points in the imaginary axis**, *without converging to anything.*

By the way, notice that if q above is a power of 2, then sooner or later we have either $\mu_n = 0$ or $\mu_n = \frac{1}{2}$, which corresponds to the sequence diverging to ∞. It is easy to see that this can only happen in this case. We conclude that **the set of initial points along the imaginary axis that lead to a sequence that "blows up" at some point, is characterized by:**

$$\mu = \frac{p}{2^m}\,, \quad \text{with } p \geq 0 \text{ and } m \geq 0 \text{ integers.}$$

Notice that this is a **dense set.**

On the other hand, **if μ is an irrational number**, *then it can be shown that the sequence of μ_n's never repeats and wonders over an infinite set of points in the unit interval. The general behavior of the sequence of μ_n's is quite complicated and gives an example of chaotic behavior. One way to see this is to write μ_0 in binary notation, i.e.:*

$$\mu_0 = 0.abcd\dots \tag{2.3.2}$$

where the a, b, c … are either zeros or ones. Multiplication by 2 in binary notation reduces to shifting the dot to the right one place. It is then clear that μ_n is obtained from μ_0 above in (2.3.2) by shifting the dot to the right n places and eliminating the digits that end up on the left of the dot. It turns out that this operation is a well understood one in the theory of Chaos and provides the simplest example of it.

By the way: the contents of note 2.3.2 is not part of the answer you were expected to supply, of course! You were only expected to think a bit about what the iterates do, but that is it.

Remark 2.3.1 *The original statement for this problem had an error: it was implied that, if μ_0 is irrational, then the sequence of angles generated gets arbitrarily close to any angle. This is actually not true, as we show next using the binary expansion for μ_0.*

The binary expansion for a rational number has a "tail" that is periodic (i.e.: after a while the sequence of numbers that makes up the binary expansion falls into a repetitive pattern), while that of an irrational number is not. Thus, pick an arbitrary irrational number ν such that $0 < \nu < 1$ and consider its binary expansion. Then select μ_0 as the number whose binary expansion is obtained from that of ν by replacing every digit by a repeated pair. For example:

$$\nu = 0.1011010001\ldots \quad \Longrightarrow \quad \mu_0 = 0.11001111001100000011\ldots$$

It is then clear that both:

- *μ_0 is irrational.*
- *The sequence of μ_n's cannot not get arbitrarily close to any number with a string 101 or 010 in its binary expansion. For example, consider $r = 0.101$. Then, since all the numbers in the sequence must have one of the forms:*

$$\begin{aligned} \mu_n &= 0.000\ldots \quad \text{or} \quad \mu_n = 0.011\ldots \quad \text{or} \\ \mu_n &= 0.100\ldots \quad \text{or} \quad \mu_n = 0.111\ldots \quad \text{or} \\ \mu_n &= 0.000\ldots \quad \text{or} \quad \mu_n = 0.001\ldots \quad \text{or} \\ \mu_n &= 0.110\ldots \quad \text{or} \quad \mu_n = 0.111\ldots, \end{aligned}$$

  it follows that their distance to r will, at best, be no less than 0.001 (in binary notation).

THE END.