Part II Problems and Solutions

Problem 1: *[Eigenvalues, eigenvectors]* **(a)** *Find the eigenvalues and eigenvectors of the companion matrix for $\ddot{x} + 4\dot{x} + 3x = 0$. On the x, y plane draw the eigenlines. For each of the two eigenlines, write down a solution which moves along it. Compare this with the work you did on the part II problem in the previous session.*

(b) *Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}$. Sketch the eigenlines, and for each eigenline write down all the solutions whose trajectories lie on that line.*

(c) *Now, invoke* `Linear Phase Portraits: Matrix Entry` *again, set $a, b, c,$ and d to display the phase plane for this matrix, and sketch the phase plane that it displays. Make sure to include (by clicking on the phase plane) at least one trajectory in each wedge between eigenlines. Include arrows indicating the direction of time.*

Solution: (a) $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$ has characteristic polynomial $p_A(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ -3 & -4 - \lambda \end{bmatrix} = -\lambda(-4 - \lambda) + 3 = \lambda^2 + 4\lambda + 3$ (the same as the characteristic polynomial of $D^2 + 4D + 3I!$), which has roots $\lambda_1 = -1$, $\lambda_2 = -3$. A vector $\mathbf{v}$ is an eigenvector for eigenvalue λ when $(A - \lambda I)\mathbf{v} = \mathbf{0}$. With $\lambda = -1$ this gives $\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \mathbf{v_1} = \mathbf{0}$, so a nonzero eigenvalue for $\lambda = -1$ is $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any nonzero multiple. With $\lambda = -3$ this gives $\begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \mathbf{v_1} = \mathbf{0}$, so a nonzero eigenvalue for $\lambda = -3$ is $\mathbf{v_2} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ or any nonzero multiple.

For the eigenline for the eigenvalue -1 the basic solution is $\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$, so the general solution along that eigenline is $c \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ for a constant c. For the eigenline for the eigenvalue -3 the basic solution is $\begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$, so the general solution along that eigenline is $c \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$ for a constant c.

These are the solutions found in **3(c)**.

(b) $p_A(\lambda) = (2 - \lambda)(-3 - \lambda) + 4 = \lambda^2 + \lambda - 2$, so the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 1$. (The order doesn't matter.) For $\lambda = -2$ a nonzero eigenvector is given by $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and

for $\lambda = 1$ by $\mathbf{v_2} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. The general solution along the first eigenline is $ce^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and along the second is $ce^t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

(c)
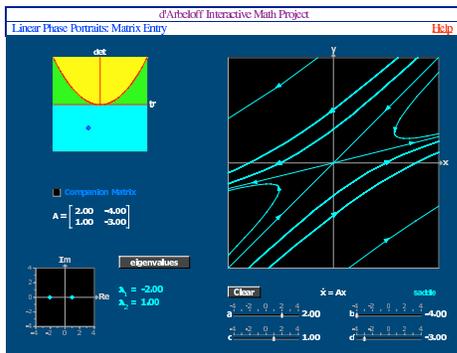
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18.03SC Differential Equations
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