18.03SC Practice Problems 22

Fourier Series

Suppose that $f(t)$ a periodic function for which 2π is a period (so that $f(t+2\pi) = f(t)$). (For convergence properties, we also assume that $f(t)$ is piecewise continuous and that $f(a) = \frac{1}{2}(f(a-) + f(a+))$ at points of discontinuity.)

Then there is exactly one sequence of numbers $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, for which

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots$$

$$+ b_1\sin(t) + b_2\sin(2t) + \cdots$$

This expansion is called the *Fourier series* for $f(t)$, and the numbers from this sequence are defined to be the *Fourier coefficients* of $f(t)$.

The Fourier coefficients of such a function $f(t)$ can be calculated directly by using integral formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt,$$

but often they can be found more easily, by starting from some known examples. One example that we will use frequently is the standard squarewave, $sq(t)$. The standard squarewave is defined to be the odd function $\mathrm{sq}(t)$ of period 2π such that $\mathrm{sq}(t) = 1$ for $0 < t < \pi$. The Fourier series for $sq(t)$ can be computed from the integral formulas to be

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right) = \frac{4}{\pi}\sum_{k\text{ odd}} \frac{\sin(kt)}{k}.$$

1. Graph the function $f(t)$ which is even, periodic of period 2π, and such that $f(t) = 2$ for $0 < t < \frac{\pi}{2}$ and $f(t) = 0$ for $\frac{\pi}{2} < t < \pi$. Find its Fourier series in two ways:

(a) Use the integral expressions for the Fourier coefficients. (Is the function even or odd? What can you say right off about the coefficients?)

(b) Express $f(t)$ in terms of $\mathrm{sq}(t)$, substitute the Fourier series for $\mathrm{sq}(t)$ and use some trig identities.

(c) Now find the Fourier series for $f(t) - 1$.

2. What is the Fourier series for $\sin^2 t$?

3. Graph the odd function $g(x)$ which is periodic of period π and such that $g(x) = 1$ for $0 < x < \frac{\pi}{2}$. 2π is also a period of $g(x)$, so it has a Fourier series of period 2π as above. Find it by expressing $g(x)$ in terms of the standard squarewave.

4. Graph the function $h(t)$ which is odd and periodic of period 2π and such that $h(t) = t$ for $0 < t < \frac{\pi}{2}$ and $h(t) = \pi - t$ for $\frac{\pi}{2} < t < \pi$. Find its Fourier series, starting with your solution to **1(c)**.

5. Explain why any function $F(x)$ is a sum of an even function and an odd function in just one way. What is the even part of e^x? What is the odd part?

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