IVP's and t-translation

1. Introductory Example

Consider the system $\dot{x} + 3x = f(t)$. In the previous note we found its unit impulse response:

$$w(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-3t} & \text{for } t > 0 \end{cases} \quad = \quad u(t)e^{-3t}.$$

This is the response from rest IC to the input $f(t) = \delta(t)$. What if we shifted the impulse to another time, say, $f(t) = \delta(t-5)$? Linear time invariance tells us the response will also be shifted. That is, the solution to

$$\dot{x} + 3x = \delta(t-2), \quad \text{with rest IC} \tag{1}$$

is

$$x(t) = w(t-2) = \begin{cases} 0 & \text{for } t < 2 \\ e^{-3t} & \text{for } t > 2 \end{cases} \quad = \quad u(t-2)e^{-3(t-2)}.$$

In words, this is a system of exponential decay. The decay starts as soon as there is an input into the system. Graphs are shown in Figure 1 below.

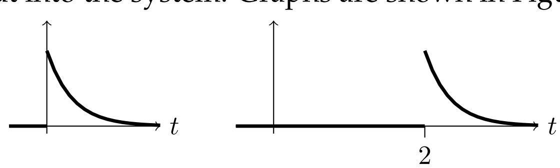

Figure 1. Graphs of $w(t)$ and $x(t) = w(t-2)$.

We know that $\mathcal{L}(\delta(t-a)) = e^{-as}$. So, we can find $X = \mathcal{L}(x)$ by taking the Laplace transform of (1).

$$(s+3)X(s) = e^{-2s} \Rightarrow X(s) = \frac{e^{-2s}}{s+3} = e^{-2s}W(s),$$

where $W = \mathcal{L}w$. This is an example of the t-translation rule.

2. t-translation Rule

We give the rule in two forms.

$$\mathcal{L}(u(t-a)f(t-a)) \quad = \quad e^{-as}F(s) \tag{2}$$

$$\mathcal{L}(u(t-a)f(t)) \quad = \quad e^{-as}\mathcal{L}(f(t+a)). \tag{3}$$

For completeness we include the formulas for

$$\begin{aligned} \mathcal{L}(u(t-a)) &= e^{-as}/s & (4) \\ \mathcal{L}(\delta(t-a)) &= e^{-as}. & (5) \end{aligned}$$

Remarks:
1. Formula (3) is ungainly. The notation will become clearer in the examples below.
2. Formula (2) is most often used for computing the inverse Laplace transform, i.e., as

$$u(t-a)f(t-a) = \mathcal{L}^{-1}\left(e^{-as}F(s)\right).$$

3. These formulas parallel the s-shift rule. In that rule, multiplying by an exponential on the time (t) side led to a shift on the frequency (s) side. Here, a shift on the time side leads to multiplication by an exponential on the frequency side.

Proof: The proof of (2) is a very simple change of variables on the Laplace integral.

$$\begin{aligned} \mathcal{L}(u(t-a)f(t-a)) &= \int_0^\infty u(t-a)f(t-a)e^{-st}\,dt \\ &= \int_a^\infty f(t-a)e^{-st}\,dt \quad (u(t-a)=0 \text{ for } t<a) \\ &= \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau \quad (\text{change of variables:} \quad \tau = t-a) \\ &= e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau \\ &= e^{-as}F(s). \end{aligned}$$

Formula (3) follows easily from (2). The easiest way to proceed is by introducing a new function. Let $g(t) = f(t+a)$, so

$$f(t) = g(t-a) \quad \text{and} \quad G(s) = \mathcal{L}(g) = \mathcal{L}(f(t+a)).$$

We get

$$\mathcal{L}(u(t-a)f(t)) = \mathcal{L}(u(t-a)g(t-a)) = e^{-as}G(s) = e^{-as}\mathcal{L}(f(t+a)).$$

The second equality follows by applying (2) to $g(t)$.

Example. Find $\mathcal{L}^{-1}\left(\dfrac{ke^{-as}}{s^2+k^2}\right)$.

Solution. $f(t) = \mathcal{L}^{-1}\left(\dfrac{k}{s^2+k^2}\right) = \sin(kt)$.

$\Rightarrow\ \mathcal{L}^{-1}\left(\dfrac{ke^{-as}}{s^2+k^2}\right) = u(t-a)f(t-a) = u(t-a)\sin k(t-a)$.

Example. $\mathcal{L}\left(u(t-3)t\right) = e^{-3s}\mathcal{L}(t+3) = e^{-3s}\left(\dfrac{1}{s^2}+\dfrac{3}{s}\right)$.

Example. $\mathcal{L}(u(t-3)\cdot 1) = e^{-3s}\mathcal{L}(1) = e^{-3s}/s$.

Example. Find $\mathcal{L}(f)$ for $f(t) = \begin{cases} 0 & \text{for } t<2 \\ t^2 & \text{for } t>2. \end{cases}$

Solution. $f(t) = u(t-2)t^2 \ \Rightarrow\ F(s) = e^{-2s}\mathcal{L}((t+2)^2) = e^{-2s}(\dfrac{2}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s})$.

Example. Find $\mathcal{L}(f)$ for $f(t) = \begin{cases} \cos(t) & \text{for } 0<t<2\pi \\ 0 & \text{for } t>2\pi. \end{cases}$

Solution. $f(t) = \cos(t)(u(t)-u(t-2\pi)) = u(t)\cos(t) - u(t-2\pi)\cos(t)$.
$\Rightarrow\ F(s) = \dfrac{s}{s^2+1} - e^{-2\pi s}\mathcal{L}(\cos(t+2\pi)) = (1-e^{-2\pi s})\dfrac{s}{s^2+1}$.

We will look at more involved examples in the next note.