18.03SC Practice Problems 9

Solutions to second order ODEs

Solution suggestions

1. *Check that both $x = \cos(\omega t)$ and $x = \sin(\omega t)$ satisfy the second order linear differential equation*

$$\ddot{x} + \omega^2 x = 0$$

This equation is called the harmonic oscillator.

If $x = \cos(\omega t)$, then $\dot{x} = -\omega \sin(\omega t)$ and $\ddot{x} = -\omega^2 \cos(\omega t) = -\omega^2 x$. If $x = \sin(\omega t)$, then $\dot{x} = \omega \cos(\omega t)$ and $\ddot{x} = -\omega^2 \sin(\omega t) = -\omega^2 x$.

2. *In fact, check that any sinusoidal function with circular frequency ω, $A\cos(\omega t - \phi)$, satisfies the equation $\ddot{x} + \omega^2 x = 0$.*

If $x = A\cos(\omega t - \phi)$, then $\dot{x} = -A\omega \sin(\omega t - \phi)$, and $\ddot{x} = -A\omega^2 \cos(\omega t - \phi) = -\omega^2 x$.

3. *Among the functions $x(t) = A\cos(\omega t - \phi)$, which have $x(0) = 0$? Doesn't this contradict the uniqueness theorem for differential equations?*

$x(0) = A\cos\phi$. When $A = 0$, then $x(t) = 0$ for every t; when $A \neq 0$, $x(0) = 0$ implies $\cos\phi = 0$, and hence ϕ can be any odd multiple of $\pi/2$. So, up to sign, the solutions that satisfy the given initial condition are $x(t) = A\cos(\omega t - \pi/2) = A\sin(\omega t)$, where $A \neq 0$ can be arbitrary.

This does not contradict the uniqueness theorem, because the uniqueness theorem as stated only applies to first order equations.

4. *Given numbers x_0 and $\dot{x}_0$, can you find a solution to $\ddot{x} + \omega^2 x = 0$ for which $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$? How many such solutions are there?*

The general solution to this differential equation is $x(t) = a\cos(\omega t) + b\sin(\omega t)$. Taking into account the given initial conditions, have $x(0) = a$, so $a = x_0$, and $x'(0) = -a\omega \sin 0 + b\omega \cos 0 = b\omega = \dot{x}_0$, so $b = \dot{x}_0/\omega$. That is, the solution that satisfies the initial conditions is $x = x_0 \cos(\omega t) + \frac{\dot{x}_0}{\omega}\sin(\omega t)$, and there is only one such solution.

5. *Let r denote a constant, which is perhaps complex valued. Suppose that e^{rt} is a solution to $\ddot{x} + kx = 0$. What does r have to be?*

Let $x = e^{rt}$. Then $\dot{x} = re^{rt}$, $\ddot{x} = r^2 e^{rt}$, and $\ddot{x} + kx = (r^2 + k)e^{rt}$. We want this to be zero. Since e^{rt} is never zero, the other factor must be zero, and $r^2 = -k$. That is, r must have the form $\pm i\sqrt{k}$ (this will be two real numbers if $k < 0$).

6. *Find a solution x_1 to $\ddot{x} - a^2 x = 0$ [note the sign!] such that $x_1(0) = 1$ and $\dot{x}_1(0) = 0$. Find another solution x_2 such that $x_2(0) = 0$ and $\dot{x}_2(0) = 1$.*

We can assume $a \geq 0$. From Question 5 we know that both $x(t) = e^{at}$ and $x(t) = e^{-at}$ are solutions to $\ddot{x} - a^2 x = 0$. Then for any constants c_1 and c_2, $x(t) = c_1 e^{at} + c_2 e^{-at}$ is also a solution to $\ddot{x} - a^2 x = 0$, with $x(0) = c_1 + c_2$ and $\dot{x}(0) = a(c_1 - c_2)$. So, for $a > 0$, $x_1(t)$ must satisfy $c_1 + c_2 = 1$ and $a(c_1 - c_2) = 0$, which implies

$c_1 = c_2 = 1/2$. So

$$x_1(t) = \frac{1}{2}e^{at} + \frac{1}{2}e^{-at} = \cosh(at).$$

For $x_2(t)$, need $c_1 + c_2 = 0$ and $a(c_1 - c_2) = 1$, so $c_1 = -c_2 = \frac{1}{2a}$ and

$$x_2(t) = \frac{1}{2a}e^{at} - \frac{1}{2a}e^{-at} = \frac{1}{a}\sinh(at).$$

If $a = 0$, $x_1(t) = 1$ and x_2 does not exist.

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