

**PROFESSOR:** Hi everyone. Welcome back. So today, we're going to take a look at some forced oscillators and the exponential response formula. And the problem we're going to take a look at is first, for part one, to consider the equation  $x'' + 8x = \cos(\omega t)$ , and then a forcing term on the right-hand side,  $\cos(\omega t)$ . And we're going to consider the case when  $\omega^2$  is not equal to 8.

So in the language of resonance, we're not on resonance. And we're also asked why is this called an undamped forced oscillator. And then for part two, to use the ERF to solve the differential equation  $x'' + 2x' + 4x = \cos(3t)$ . And then, what is the natural angular frequency of this differential equation? So I'll let you take a look at these problems and try them out for yourself. And I'll be back in a moment.

Hi everyone. Welcome back. So we're asked to find the general solution to the differential equation,  $x'' + 8x = \cos(\omega t)$ . And notice how this is a differential equation with constant coefficients. But it's being forced by a periodic function on the right-hand side.

So the first thing to do is to write down the homogeneous solution. So the homogeneous solution, which I'll denote with a subscript  $h$ , solves the differential equation with the right-hand side of 0. This gives us a characteristic polynomial of  $s^2 + 8 = 0$ , which then gives us roots of  $\pm \sqrt{8}i$ .

So the homogeneous solution is some constant  $c_1$ . And when we have purely imaginary roots, we have  $\cos(\text{imaginary term} \times t)$ , plus some constant  $c_2$  times  $\sin(\text{imaginary term})$ , which in this case is  $\sqrt{8}t$ . So the homogeneous solution is always contains two constants. And it solves the differential equation with a zero right-hand side.

The general solution, however, is the homogeneous solution plus one particular solution that solves the differential equation  $x'' + 8x = \cos(\omega t)$ . So we just need to find one solution to this differential equation. And in this case, I'll use the exponential response formula. But first note that when we use the exponential response formula, we need a forcing to be of the form  $e^{st}$  on the right-hand side. And  $\cos(\omega t)$  is not in that form. However, what we can do is there's a trick to complexify the right-hand side. So we note that the  $\cos(\omega t)$  is actually the real part of  $e^{i\omega t}$ .

So what this means is if we have a complex solution,  $z$ , which solves  $z'' + 8z = e^{i\omega t}$ , then we can take  $x$  equals the real part of  $z$ , then solves the differential equation  $x'' + 8x = \cos \omega t$ . And now we're in business because this equation has the form where we can use the exponential response formula.

So solving this differential equation for  $z$ , we have a particular solution for  $z$ , is going to be  $e^{i\omega t}$  divided by the characteristic polynomial evaluated at the exponential. So in this case, the exponential is  $i\omega$ . So we must evaluate the polynomial at  $i\omega$ . And this is the solution provided that the polynomial evaluated at  $i\omega$  doesn't vanish. And in our case, the characteristic polynomial,  $p$  of  $s$ , we worked out already to be  $s^2 + 8$ .

So this gives us a particular solution for  $z$ , which is  $1$  divided by  $i\omega^2 + 8$  on the denominator. We have  $e^{i\omega t}$  on the numerator. And of course,  $1$  over  $i\omega^2 + 8$  becomes  $1$  over  $8 - \omega^2$ . And we know we don't have a problem because  $\omega$ , we were told in the problem, is not equal to the square root of  $8$ . So in this case, we know that the denominator is not going to vanish.

And now what we need to do is just take  $x$ , for the particular solution, to be the real part of  $z$ , which is going to be the real part-- I'll write it out--  $\frac{1}{8 - \omega^2} \cos \omega t + i \frac{1}{8 - \omega^2} \sin \omega t$ , which gives us  $\frac{1}{8 - \omega^2} \cos \omega t$ . So the total solution, the general solution, is going to be the sum of the homogeneous solution plus the particular solution. And in our case, that's going to be  $c_1 \cos \omega t + c_2 \sin \omega t + \frac{1}{8 - \omega^2} \cos \omega t$ . Sorry. These should be root  $8$ 's.

So this is the homogeneous part. And this is the particular solution. And this is the general solution. So we were asked, also, why this is sometimes called an undamped forced oscillator. Well, it's undamped, because in the differential equation, there is no damping term. There's no term proportional to  $x'$ .

And then, secondly, it's forced because we have a forcing on the right-hand side. We have an input function,  $f$ , which in this case is  $\cos \omega t$  which doesn't depend on  $x$ . And that forces the differential equation. Also note that the forcing term gives rise to part of the solution which is directly proportional to the forcing. In fact, it has the same frequency but a different amplitude. So this concludes part A.

Now for part B, we're asked to use the ERF to solve the differential equation  $x'' + 2x' + 4x = \cos 3t$ . And again, we can use the same trick. The right-hand side isn't of the form  $e^{i3t}$ . But what we can do is we can write  $\cos 3t$  as the real part of  $e^{i3t}$ , and then solve the differential equation  $z'' + 2z' + 4z = e^{i3t}$ , and then take  $x = \text{Re}(z)$ .

And in this case, we're only looking for a particular solution, which we can compute using the ERF. So it's  $1$  over the characteristic polynomial evaluated at  $3i$  times  $e^{i3t}$ . And what's a characteristic polynomial of this differential equation? Well,  $p(s)$  is going to be  $s^2 + 2s + 4$ , which means that  $p(3i)$  is going to be  $(3i)^2 + 2(3i) + 4$ , which gives us  $-9 + 6i + 4 = -5 + 6i$ .

And putting the pieces together, we end up getting that  $x$  is equal to the real part of  $\frac{1}{-5 + 6i} e^{i3t}$ . And we can expand out the numerator using Euler's formula to get  $\cos 3t + i \sin 3t$ . And when the dust settles, I got  $\frac{1}{61} (-5 \cos 3t + 6 \sin 3t)$ . So I'll let you work at this last step for yourself.

And then, lastly, what is the natural angular frequency of the differential equation? Well, this is just some notation. We sometimes call the natural angular frequency to be the square root of the term proportional to  $x$ . So in this case, the term two here comes in as a damping term. The natural frequency, which is sometimes written as  $\omega^2$ , is always equal to this term, which is  $4$ . So the natural frequency, the natural angular frequency,  $\omega$ , would be the square root of  $4$ , which would be  $2$ .

So I just like to quickly recap. We've taken a look at forced oscillators. And we computed their solutions using the ERF. And one common trend, particularly when we have an oscillating input, a forcing term, is to first change the oscillating term into a complex exponential, then compute a particular solution using the ERF formula, and then take the real part of the ERF solution to recover a real solution to the ODE. That gives us the particular solution. And then, in some cases, we also have to add the homogeneous solution to it to get the full general solution to the ODE. So I'd like to conclude here. And I'll see you next time.