Superposition

1. Superposition I

We saw the principle of superposition already, for first order equations. For example, we saw that if y_1 is a solution to $y' + 4y = \sin(3t)$ and y_2 a solution to $y' + 4y = 2$, then $y_1 + y_2$ is a solution to $y' + 4y = \sin(3t) + 2$. Superposition will be useful for us again, though now we will use it in two slightly different ways. The first version we already used in a previous session, but let's state it carefully and explicitly:

Superposition I: If y_1 and y_2 are solutions of a *homogeneous* linear equation, then so is any linear combination; that is, for any constants c_1 and c_2, the function $y_3 = c_1 y_1 + c_2 y_2$ will also be a solution.

Example. Consider the ODE

$$t^2 y'' + t y' - 4y = 0.$$

This is homogeneous, since the constant term (the one not involving y or any of its derivatives) is zero. You can easily check by substitution that $y_1(t) = t^2$ and $y_2(t) = 1/t^2$ are both solutions. Thus

$$y(t) = c_1 t^2 + c_2/t^2$$

is a solution for any c_1 and c_2.

Notice that we didn't need the differential equation to have *constant coefficients*: linearity and homogeneity is enough.

If the equation is of second order with two solutions y_1 and y_2 such that neither is a multiple of the other, then

$$c_1 y_1 + c_2 y_2$$

will be the *general* solution. It has the right number of parameters. The restriction on the solutions is to make sure that they are really "different" solutions, for instance, in the above example, it would be incorrect to take $y_1 = t^2$ and $y_2 = 3t^2$, and then claim that

$$y(t) = c_1 t^2 + c_2 \cdot 3t^2 = (c_1 + 3c_2) t^2$$

is the general solution.

2. Superposition II

Now consider the linear second order equation

$$mx'' + bx' + kx = F_{ext}(t), \tag{1}$$

and its associated homogeneous equation

$$mx'' + bx' + kx = 0. \tag{2}$$

Superposition II: Suppose x_p is any solution to (1). If x_h is any solution to (2), then $x = x_p + x_h$ is again a solution to (1).

This is similar to the way we used superposition for first order equations. To prove this, we just need to substitute x into (1) and check that it really is a solution:

$$\begin{aligned} mx'' + bx' + kx &= m(x_h + x_p)'' \ + \ b(x_h + x_p)' \ + \ k(x_h + x_p) \\ &= (mx_h'' + mx_p'') \ + \ (bx_h' + bx_p') \ + \ (kx_h + kx_p) \\ &= (mx_h'' + bx_h' + kx_h) \quad + \quad (mx_p'' + bx_p' + kx_p) \\ &= \qquad\qquad 0 \qquad\qquad + \qquad\quad F_{ext}. \end{aligned}$$

So indeed, it is a solution.

An important fact: if x_h is the *general* solution to (2) (so it should have two parameters) then $x_p + x_h$ is the general solution to (1). We'll see an example of this shortly.

This proof works for linear equations of any order. For example, we already saw it as a consequence of the method of integrating factors for first order equations.

We've already seen how to find the general solution to the associated homogeneous equation (2) using the characteristic equation. Thus to find the general solution to (1), we simply need to do is find a *single* solution to this particular equation. This is what we'll discuss next.

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