18.03SC Practice Problems 11

Exponential and sinusoidal input signals

Solution suggestions

1. *Find A so that $A\sin(3t)$ is a solution of $\ddot{x} + 4x = \sin(3t)$. What is the general solution?*

Plug the desired solution into the differential equation to solve for A. If $x = A\sin(3t)$, then $\ddot{x} = -9A\sin(3t)$, and A must satisfy $-5A\sin(3t) = \ddot{x} + 4x = \sin(3t)$, so $A = -1/5$. Therefore, $x_p(t) = (-1/5)\sin(3t)$ is a solution of the given equation.

The general solution is the sum of a particular solution and the homogeneous solution - the general solution to the homogeneous equation $\ddot{x} + 4x = 0$. The characteristic polynomial here is $p(s) = s^2 + 4$, with roots $\pm 2i$, so the general homogeneous solution is $x_h = C_1\sin(2t) + C_2\cos(2t)$. Therefore, the general solution to $\ddot{x} + 4x = \sin(3t)$ is given by

$$x = x_p + x_h = -\frac{1}{5}\sin(3t) + C_1\sin(2t) + C_2\cos(2t)$$

for arbitrary real constants C_1 and C_2.

2. *For $\omega \geq 0$, find A such that $A\cos(\omega t)$ is a solution of $\ddot{x} + 4x = \cos(\omega t)$. Graph the input signal $\cos(\omega t)$ and the solution $A\cos(\omega t)$ for $\omega = 0$, $\omega = 1$, and $\omega = 3$. Sketch a graph of A as a function of ω, as ω ranges from 0 to 5. Where does resonance occur? What is the significance of the sign of A?*

If $x = A\cos(\omega t)$, then taking derivatives gives us $\ddot{x} = -\omega^2 A\cos(\omega t)$, and $\ddot{x} + 4x = (4 - \omega^2)A\cos(\omega t)$. So for the differential equation to be satisfied, $A = \frac{1}{4-\omega^2}$.

Using this formula for amplitude gain we can graph input and output signals for the requested values of ω. When $\omega = 0$, $A = 1/4$, the input is the constant 1 and the output (solution) is also constant $1/4$; when $\omega = 1$, $A = 1/3$, the input is $\cos(t)$ and the output is $(1/3)\cos(t)$; when $\omega = 3$, $A = -1/5$, the input is $\cos(3t)$ and the solution is $(-1/3)\cos(3t)$.

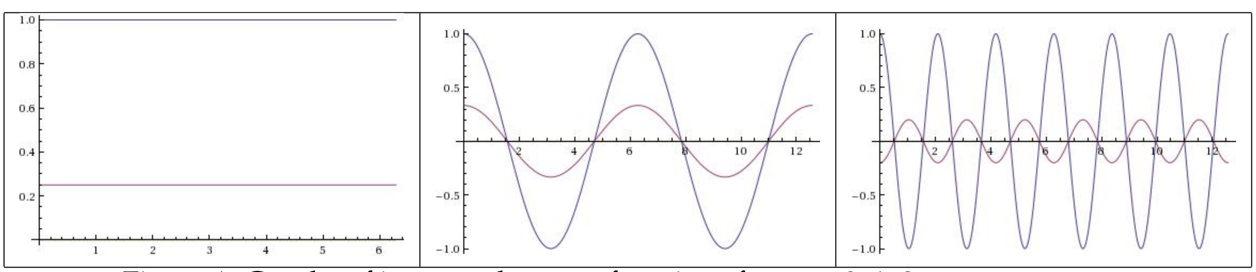

Figure 1: Graphs of input and output functions for $\omega = 0, 1, 3$.

Here is the graph of the amplitude gain itself, $A = A(\omega) = 1/(4 - \omega^2)$.

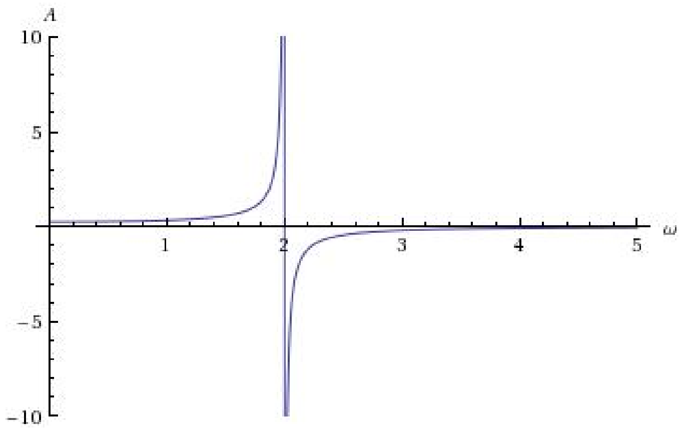

Figure 2: Graph of $A = A(\omega) = 1/(4-\omega^2)$.

The graph of $A(\omega)$ in the range of $\omega \in [0,5]$ has two branches: when $0 \le \omega < 2$, $A(\omega)$ increases from $1/4$ to positive infinity; when $2 < \omega \le 5$, $A(\omega)$ increases from negative infinity to $-\frac{1}{21}$.

Resonance occurs at the pole – when $\omega = 2$. $A > 0$ means in phase, $A < 0$ means antiphase, i.e., 180 degrees out of phase.

3. *Find an exponential solution of* $\dfrac{d^4x}{dt^4} - x = e^{-2t}$.

The characteristic polynomial of the homogeneous equation is given by $p(s) = s^4 - 1$. Since $p(-2) = 15 \neq 0$, the exponential response formula gives the solution $x = \frac{e^{-2t}}{p(-2)} = \frac{1}{15}e^{-2t}$.

4. *Find a sinusoidal solution of* $\dfrac{d^4x}{dt^4} - x = \cos(2t)$.

We pass to the complex world and solve the corresponding complex differential equation $\dfrac{d^4z}{dt^4} - z = e^{i2t}$ – the same system driven by the complex-valued exponential input function with real part $\cos(2t)$. Since the characteristic polynomial for this system is $p(s) = s^4 - 1$ and $p(2i) = 15 \neq 0$, the exponential response formula yields the complex-valued solution $z = \frac{1}{15}e^{2it}$ to the new equation. A sinusoidal solution to the original equation is then given by the real part, $x = \mathrm{Re}(z) = \frac{1}{15}\cos(2t)$.

5. *Find the general solution of the differential equations in* ***(3)*** *and* ***(4)****.*

To get the general solution, we take the sum of the general solution to the homogeneous equation and the particular solution to the original equation. The homogeneous equation corresponding to both (3) and (4) is $\frac{d^4x}{dt^4} - x = 0$. The characteristic polynomial $p(s) = s^4 - 1$ has four roots: ± 1, $\pm i$. So the general solution to $\frac{d^4x}{dt^4} - x = 0$ is given by $C_1e^t + C_2e^{-t} + C_3\cos(t) + C_4\sin(t)$ for arbitrary real constants C_1, C_2, C_3 and C_4.

Therefore, the general solution to the equation in (3) is $\frac{1}{15}e^{-2t} + C_1e^t + C_2e^{-t} + C_3\cos(t) + C_4\sin(t)$.

The solution to the equation in (4) is $\frac{1}{15}\cos(2t) + C_1e^t + C_2e^{-t} + C_3\cos(t) + C_4\sin(t)$.