# 18.03SC Practice Problems 5

## First order Linear ODEs: Integrating factors

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Reduced standard form: $\dot{x}(t) + p(t)x(t) = q(t)$, briefly $\dot{x} + px = q$     $(*)$

Integrating factor: a function $u(t)$ such that $u\dot{x} + upx = \dfrac{d}{dt}(ux)$.

Computation: Thus $pu = \dot{u}$, or $u = e^{\int p(t)\,dt}$

Multiply $(*)$ by $u$: $\dfrac{d}{dt}(ux) = uq$, so $x = u^{-1} \displaystyle\int uq\,dt$

The general solution of $(*)$ has the form $x = x_p + cx_h$ where $x_h = u^{-1}$ is a nonzero solution of the homogeneous equation $\dot{x} + px = 0$.

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**1.** Around here, the ocean experiences tides. About twice a day the ocean level rises and falls by several feet. This is why small boats are often tied up to floating docks.

A salt pond on Cape Cod is connected to the ocean by means of a narrow channel. This problem will explore how the water level in the pond varies.

In roughest terms, the water level in the bay increases, over a small time interval, by an amount which is proportional to (1) the difference between the ocean level and the bay level and (2) the length of the small time interval.

**(a)** Write $y(t)$ for the height of the water in the ocean, measured against some zero mark, and $x(t)$ for the height of the water in the bay, measured against the same mark. Set up the first order linear equation that describes this model. What is the "system" here? What part of the ODE represents it? What function is the "input signal"? What is the "output signal"?

**(b)** Assume that the tide is high exactly every $4\pi$ hours – not a bad approximation. Suppose that the ocean height is given by $y(t) = \cos(\omega t)$ (in meters and hours). What value does $\omega$ take?

**(c)** Find a solution of your differential equation using integrating factors. You may find the following integral useful – a consequence, you may recall, of two integration by parts:

$$\int e^{kt}\cos(\omega t)\,dt = \frac{1}{k^2 + \omega^2}e^{kt}(k\cos(\omega t) + \omega\sin(\omega t)) + c$$

**(d)** Your solution probably had the form $a\cos(\omega t) + b\sin(\omega t)$ for some constants $a$, $b$. Find it a second time by subsituting this expression into the differential equation and solving for $a$ and $b$.

**2.** In a previous session we studied the direction field of the differential equation

$$\frac{dy}{dx} = x - 2y.$$

Now find the general solution of this equation analytically.

When we studied the direction field we found a straight line solution. Does it occur within the general solution here?

Plot it and several other solutions, using your expression for the general solution. Is there a funnel?

Find the particular solution with initial value $y(0) = 1$.

**3.** Find the general solution of

$$x^2 \frac{dy}{dx} + 2xy = \sin(2x)$$

by recognizing the left hand side as the derivative of a product.

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