**DAVID SHIROKOFF:**
Hi, everyone. So today I'd like to take a look at linear equations, linear first-order differential equations. And specifically, we're going to look at just exactly, what is a linear equation? So for example, which of these equations are linear? And then second of all, we're going to take a look at superposition. And we ask to consider the differential equation y dot plus y squared equals some function q of t. And just to demonstrate that this equation does not satisfy the superposition principle. So I'll let you take a look at this problem, and try and work it out for yourself. And I'll be back in a minute.

Hi, everyone. Welcome back. All right. So we're asked to figure out which of these equations are linear. So just to recap, what does it mean for an equation to be linear? What does it mean for a first-order equation to be linear?

Well, it means that we can write it in the general form, so if it's a function of x, dy/dx plus some general function of x. And this could be anything. But the point is that it's multiplied just by y. And then it can equal some arbitrary function q of x on the right-hand side. So every first-order linear equation can be rewritten in this form.

So let's tackle part A. All right, y dot plus k*y equals-- or y dot equals k*y. We can just rewrite it as y dot equals negative k*y. Note that just some constant k is a function of time. It's a constant function of time, but it can just be thought of as a function of time. Same with the right-hand side, it's 0. And so this equation has the same form as our general first-order linear equation. So this is a linear equation.

Let's take a look at question 2, part 2. We have y dot plus y squared t. I'll bring the y to the left-hand side. If we do that, we get this. And just by comparing this to our generic first-order linear ODE, we note that it's not of this form. So specifically, this is the bad term in question.

We see that it's y squared times t. Notice how we're not allowed to have any term that's y squared in our first-order linear equation. Now I will say this just as an aside. This equation's also known as a Bernoulli equation. And as an aside, if you were to make a substitution u equals 1 over y, and rewrote this equation in terms of u as the independent variable, you'd find out that this equation would be linear in terms of u.

But as it's written now in terms of y, it's not linear. So this is a kind of an interesting point. That sometimes we can get lucky and do a nonlinear transformation and convert a nonlinear

equation into a linear one.

So for part 3 we have y prime plus cos(x)*y equals x cubed. So again, we see that this is a linear equation because we can identify p of x with cosine x, q of x with x cubed. So this equation has the general form of a linear ODE.

For part 4, well if we take a look at it right away, y dot divided by y is equal to t squared. It doesn't appear to be linear. But of course, we can multiply through by y and rewrite things. And of course, this equation is equivalent to y dot minus t squared y equals 0. Which again, is linear because p of t can be identified with negative t squared and q of t can be identified with 0.

And then lastly, for part 5, we have x squared y y prime plus 4x equals x cubed. And again, if we were to try to write it in the general form of a linear ODE, which we have here, it would look like... 4x divided by x squared y minus x cubed divided by x squared y equals 0. And because there's a y on the denominator here, this equation doesn't have the general form of a first-order linear equation.

Now again, I note that if you were to make a substitution, u equal to y squared, that substitution would make this equation linear. So this concludes part a.

So for part b, we're given a differential equation y dot plus y squared is equal to q of t. And we want to show that this equation doesn't satisfy the superposition principle.

Now, you might note that this equation is not linear. And second of all, we know that linear equations satisfy the superposition principle. So we need to exploit the fact that it's a nonlinear equation to make it fail the superposition principle.

And just as an example, what we can do is I can just pick a couple right-hand sides q as examples of this ODE. So we can imagine we have a solution y_1 which solves the ODE y_1 dot plus y_1 squared equals-- I'm just going to say 1 on the right-hand side. And I can also take another function y_2, which say satisfies this differential equation, say t. So all I've done is I've just picked one q_1 on the right-hand side and another function q_2. And I imagined that I have a solution y_1 which satisfies this differential equation and y_2 which satisfies this differential equation.

So what does the superposition principle say? Well, superposition says that if y_1 solves this

equation and $y_2$ solves this equation, then the function $y$ equals $y_1$ plus $y_2$ must solve the equation $\dot{y}$ plus $y$ squared equals the sum of the right-hand side, $1$ plus $t$.

So if we try and substitute in $y_1$ plus $y_2$ into this equation, we're going to come to a contradiction. So let's do that now.

So we take $\frac{d}{dt}$ of $y_1$ plus $y_2$, and I'm going to add it to $y_1$ plus $y_2$ quantity squared. And now what I can do is I can just simplify this. So I get $\dot{y}_1$ plus $\dot{y}_2$. And I can expand out this square. So I get $y_1$ squared plus $y_2$ squared plus $2y_1 \cdot y_2$.

And now what I'm going to do is I'm going to combine the term $y_1$ with $y_1$ squared. And by construction, $\dot{y}_1$ plus $y_1$ squared is equal to $1$, because I assumed that it satisfied this differential equation.

And then in addition by construction, $\dot{y}_2$ plus $y_2$ squared satisfies the right-hand side of the second equation with $t$. So I get $1$ plus $t$, but I'm left over with this other piece $2y_1 \cdot y_2$. So in general, if I take a function which solves this differential equation and another function which solves this differential equation, and I add them together, and plug it into the left-hand side of this differential equation, I get something which is $1$ plus $t$ plus some other stuff. And if it did satisfy the principle of superposition, it must equal $1$ plus $t$. So we arrive at a contradiction because it doesn't equal $1$ plus $t$.

In fact, it fails to equal $1$ plus $t$ by this term $2y_1 \cdot y_2$. And this term comes directly from the fact that we had a nonlinearity in the equation. So this is just one illustration of the fact that a nonlinear equation doesn't necessarily satisfy the superposition principle. Whereas, every linear equation satisfies the superposition principle. This is one reason we love linear equations and we study them extensively.

OK, so I'd just like to conclude here. And I'll see you next time.