Examples of Constant Coefficient First order Equations

We have seen a number of physical systems that are modeled by first order constant coefficient linear ODE's. Here we will recall some of them and introduce examples of radioactive decay and of mixing tanks.

Example 1. Radioactive Decay. Suppose we have some radioactive matter with *decay constant* k_1. This means that the rate at which the radioactive material decays is proportional to the amount present at any given time. Let $A(t)$ be the amount of matter at time t. The rate equation modeling this system is

$$\frac{dA}{dt} = -k_1 A(t) \quad \Leftrightarrow \quad \frac{dA}{dt} + k_1 A = 0.$$

The solution to this is easily seen to be $A(t) = A_0 e^{-k_1 t}$.

Now, suppose that when A decays it becomes a different radioactive substance B, which has its own decay constant k_2. The rate equation for the amount $B(t)$ of B is

$$\frac{dB}{dt} = k_1 A(t) - k_2 B(t),$$

i.e. $\frac{dB}{dt}$ = (rate B is created from A) - (rate B decays). Using the solution for $A(t) = A_0 e^{-k_1 t}$ this becomes

$$\frac{dB}{dt} + k_2 B(t) = k_1 A_0 e^{-k_1 t}.$$

Example 2. Temperature-Concentration or Conduction-Diffusion. Newton's law of cooling gives the equation

$$\frac{dT}{dt} + kT = kT_e(t).$$

Here, $T(t)$ is the temperature of a body and $T_e(t)$ is the temperature of the surrounding environment. (See the heat-diffusion example in session 5.)

Example 3. Mixing Tanks. A tank containing a volume V of salt solution has solution entering the tank at rate r (in, say, liters/minute). The incoming solution has a concentration $C_e(t)$ of salt. Solution leaves the tank at the same rate r. (This means the volume of solution in the tank is constant –we say the inflow and outflow rates are *balanced*.)

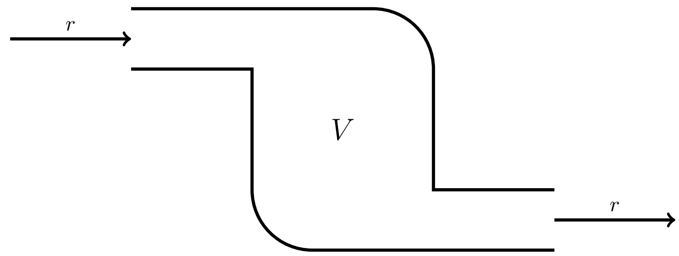

Fig. 1. A mixing tank with balanced inflow and outflow.

We will assume the concentration of salt stays uniform throughout the tank. If the solution is continuously stirred this is a reasonable assumption.

Question. Let $x(t)$ be the *amount* of salt in the tank. How do we write a differential equation modeling $x(t)$?

Answer. The rate of change of salt in the tank is

the rate salt flows in - the rate it flows out.

rate in = inflow rate $\times$ inflow concentration = $r\,C_e(t)$.
rate out = outflow rate $\times$ ouflow concentration = $r\,\frac{x}{V}$, since the outflow concentration is the concentration in the tank which is x/V.
Therefore,

$$\frac{dx}{dt} = rC_e(t) - r\frac{x}{V} \quad \Leftrightarrow \quad \dot{x} + \frac{r}{V}x = rC_e(t). \tag{1}$$

Notes: 1. In building the model it is best to let the dependent variable be the amount of salt and *not* the concentration. One good reason for this is that amounts add, but concentrations do not. For example, if I combine a solution with 2 grams of salt and one with 3 grams, I will have a solution with 5 grams of salt. But, if I combine a 2 g/liter solution with a 3 gm/liter solution, the new solution will have concentration somewhere between 2 and 3 g/liter, depending on how much of each solution is combined.

2. If we choose we can write the DE in terms of the concentration $C(t) = x(t)/V$. Simply divide the equation by V:

$$\frac{\dot{x}}{V} + \frac{r}{V}\frac{x}{V} = \frac{r}{V}C_e(t) \;\Rightarrow\; \dot{C} + \frac{r}{V}C(t) = \frac{r}{V}C_e(t).$$

If we let $k = r/V$ this DE becomes $\dot{C} + kC = kC_e$, which looks like the conduction-diffusion equation in example 2.

3. If the tanks are not balanced we can use the same logic to build the ODE

modeling the mixing tanks. It becomes the linear DE

$$\dot{x} + \frac{r_2}{V} x = r_1 C_e,$$

where r_1, r_2 are the inflow and outflow rates respectively. We can also write this in terms of concentration $\dot{C} + \frac{r_2}{V} C = \frac{r_1}{V} C_e$.

Example 4. RC Circuits In session 4 we discussed a cicuit with a resistor, capacitor and input voltage. It satisfies the ODE

$$R \frac{dI}{dt} + \frac{1}{C} I = \mathcal{E}'. \quad (2)$$

Here, R is the resistance, C is the capacitance, $\mathcal{E}$ is the voltage source and I is the current through the resistor.

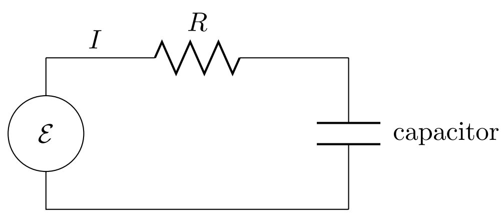

Fig. 2. RC circuit with input voltage $\mathcal{E}$.

Let $q(t)$ be the charge on the capacitor, then $I = \frac{dq}{dt}$ and equation (2) can be written in terms of q as

$$R \frac{dq}{dt} + \frac{1}{C} q = \mathcal{E}(t). \quad (3)$$

We'll take a moment to remind you that the right-hand side of the DE is not always exactly the same as the input. In both (2) and (3) we consider $\mathcal{E}$ to be the input to the system, but in (2) the right-hand side of the equation is $\mathcal{E}'$.

18.03SC Differential Equations
Fall 2011