18.03SC Unit 1 Exam

1. (a) In a perfect environment, the population of Norway rat that breeds on the MIT campus increases by a factor of $e \simeq 2.718281828459045\ldots$ each year. Model this natural growth by a differential equation. [8]

What is the growth rate k?

(b) MIT is a limited environment, with a maximal sustainable Norway rat population of $R = 1000$ rats. Write down the logistic equation modeling this. (You may use "k" for the natural growth rate here if you failed to find it in **(a)**.) [4]

(c) The MIT pest control service intends to control these rats by killing them at a constant rate of a rats per year. If it wants to limit the rat population to 75% of the maximal sustainable population, what rate a it should aim for (in rats per year)? [8]

2. For the autonomous equation $\dot{x} = x(x-1)(x+2)$, please sketch:

(a) the phase line, identifying the critical points and whether they are stable, unstable, or neither. [4]

(b) at least one solution of each basic type (so that every solution is a time-translate of one you have drawn) [4]

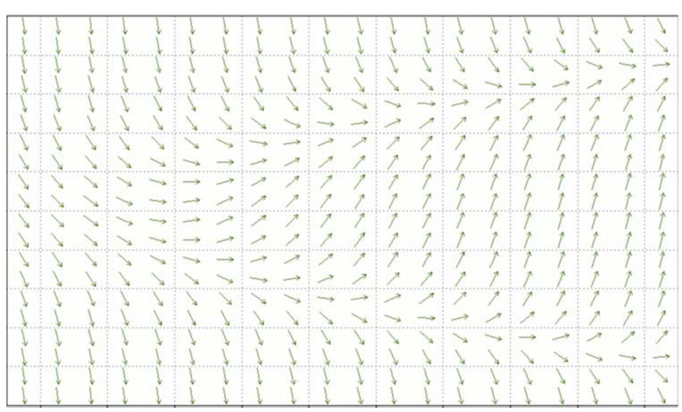
Below is a diagram of a direction field of the differential equation $y' = (1/4)(x - y^2)$. On it please plot and label:

(c) the nullcline [3]

(d) at least two quite different solutions [3]

(e) the separatrix (if there is one) [3]

(f) True or false: If $y(x)$ is a solution with a minimum, then for all large enough x, $y(x) < \sqrt{x}$. (No explanation needed: just circle one.) [3]



3. (a) Use Euler's method with stepsize $h = 1/2$ to estimate the value at $x = 3/2$ of the solution to $y' = x + y$ such $y(0) = 1$. [10]

(b) Find the solution of $t\dot{x} + x = \cos t$ such that $x(\pi) = 1$. [10]

4. (a) Find real a, b such that $\dfrac{1}{3+2i} = a + bi$. [3]

(b) Find real r, θ such that $1 - i = re^{i\theta}$. [3]

(c) Find real a, b such that $(1-i)^8 = a + bi$. [3]

(d) Find real a, b such that $b > 0$ and $a + bi$ is a cube root of -1. [3]

(e) Find real a, b such that $e^{\ln 2 + i\pi} = a + bi$. [3]

(f) Write $f(t) = 2\cos(4t) - 2\sin(4t)$ in the form $A\cos(\omega t - \phi)$. [5]

5. (a) Find a particular solution to the equation $\dot{x} + 3x = e^{2t}$. [5]

(b) Find the solution to the same equation such that $x(0) = 1$. [5]

(c) Write down a linear equation with exponential right hand side of which $\dot{x} + 3x = \cos(2t)$ is the real part. [5]

(d) Find a particular solution to the equation $\dot{x} + 3x = \cos(2t)$. [5]