18.034 SOLUTIONS TO PRACTICE FINAL EXAM, SPRING 2004

The final exam will be held on Thursday, May 20, 9:00AM–12:00NOON. The final exam will be closed notes, closed book, and calculators will not be permitted. Scratch paper and a stapler will be available. A short list of Laplace transforms will be provided (the same as the list on Exam 3). The following problems are representative of the problems on the exam.

Problem 1 Two different chemical solutions are pumped into a container of volume 100 L, each at the rate of 5 L/s, and thoroughly mixed solution is pumped out at the rate of 10 L/s. The inflow concentration of Chemical 1 is q_1 and the inflow concentration of Chemical 2 is q_2. Denote the mass of Chemical 1 by x_1 and the mass of Chemical 2 by x_2. A catalyst in the container transforms Chemical 1 into Chemical 2 at a rate $0.4x_1$ per second.

(a) Determine the 2×2 linear system of 1^{st} order ODEs that x_1 and x_2 satisfy.

Solution: For x_1 there are three contributions to x_1'. Chemical flows in at a rate $5q_1$. Chemical flows out at a rate $-10 \times (x_1/100)$, i.e. $-0.1x_1$. And chemical is catalyzed at a rate $-0.4x_1$. For x_1, chemical flows in at a rate $5q_1$, chemical flows out at a rate $-10 \times (x_2/100) = -0.1x_2$, and Chemical 1 is catalyzed into Chemical 2 at a rate $+0.4x_1$. Therefore the differential equations are,

$$\begin{cases} x_1' &= -0.5x_1 + 5q_1, \\ x_2' &= 0.4x_1 + (-0.1)x_2 + 5q_2 \end{cases}$$

In matrix form, this is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -0.5 & 0 \\ 0.4 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5q_1 \\ 5q_2 \end{bmatrix}.$$

(b) Without finding the general solution, determine the steady-state solution for x_1 and x_2.

Solution: In the steady-state, $x_1' = 0$ and $x_2' = 0$. Solving the equations gives, $0.5x_1 = 5q_1$ and $0.1x_2 = 0.4x_1 + 5q_2$. The solution is,

$$\begin{cases} x_1 &= 10q_1, \\ x_2 &= 40q_1 + 50q_2 \end{cases}$$

Problem 2 Consider the following inhomogeneous linear 1^{st} order ODE on the interval $t > 1$.

$$y' + \frac{1}{t^2 - 1} y = \sqrt{\frac{t+1}{t-1}}.$$

(a) Let $a > 0$ be a real number, and let $u(t)$ be the function,

$$u(t) = \left(\frac{t-1}{t+1} \right)^a.$$

Compute $u'/u = \frac{d}{dt} \ln(u(t))$.

Solution: By definition,

$$\ln(u(t)) = a \ln(t-1) - a \ln(t+1).$$

Therefore,

$$u'(t)/u(t) = \frac{d}{dt} \ln(u(t)) = \frac{a}{t-1} - \frac{a}{t+1} = \frac{2a}{t^2 - 1}.$$

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(b) Find a value of a such that $u(t)$ is an integrating factor.

Solution: If $u(t)$ is an integrating factor, then

$$u'(t) = \frac{1}{t^2-1}u(t).$$

By part (a), this holds if $a = \frac{1}{2}$. Therefore an integrating factor is,

$$u(t) = \sqrt{\frac{t-1}{t+1}}.$$

(c) Find the general solution of the ODE.

Solution: Multiplying both sides of the equation by an integrating factor yields,

$$\sqrt{\frac{t-1}{t+1}}y' + \frac{1}{t^2-1}\sqrt{\frac{t-1}{t+1}}y = 1, \quad \left(\sqrt{\frac{t-1}{t+1}}y\right)' = 1.$$

Integrating gives,

$$\sqrt{\frac{t-1}{t+1}}y = t + C, \quad y(t) = \sqrt{\frac{t+1}{t-1}}(t+C).$$

(d) Find the unique solution $y(t)$ such that the limit,

$$\lim_{t\to 1^+} y(t),$$

exists and is bounded.

Solution: If $C = -1$, then the solution for $t > 1$ is,

$$y(t) = \sqrt{\frac{t+1}{t-1}}(t-1) = \sqrt{t^2-1}.$$

As $t \to 1^+$, this is bounded and the limit equals 0.

Problem 3 A basic solution pair of the homogeneous linear 2^{nd} order ODE,

$$y'' + \frac{2t}{t^2-1}y' - 16\frac{1}{(t^2-1)^2}y = 0,$$

is given by $\{y_1, y_2\}$,

$$y_1(t) = \left(\frac{t-1}{t+1}\right)^2, \quad y_2(t) = \left(\frac{t+1}{t-1}\right)^2.$$

(a) Compute the Wronskian $W[y_1, y_2](t)$.

Solution: By part (a) from Problem 2,

$$y_1'(t)/y_1(t) = \frac{4}{t^2-1}, \quad y_2'(t)/y_2(t) = \frac{-4}{t^2-1}.$$

Therefore the Wronskian determinant is,

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ \frac{4}{t^2-1}y_1(t) & \frac{-4}{t^2-1}y_2(t) \end{vmatrix} = \frac{-8}{t^2-1}y_1(t)y_2(t).$$

But of course, $y_1(t)y_2(t) = 1$. Therefore the Wronskian is,

$$W[y_1, y_2](t) = \frac{-8}{t^2-1}.$$

(b) Use variation of parameters to find a particular solution of the inhomogeneous ODE,

$$y'' + \frac{2t}{t^2-1}y' - 16\frac{1}{(t^2-1)^2}y = t^2 - 1.$$

Solution: The Green's kernel for variation of parameters is,

$$K(s,t) = (y_1(s)y_2(t) - y_1(t)y_2(s))/W[y_1,y_2](s) = -\frac{1}{8}(s^2-1)(y_1(s)y_2(t) - y_1(t)y_2(s)).$$

Therefore the integrand for variation of parameters is,

$$K(s,t)f(s) = -\frac{1}{8}(s^2-1)^2(y_1(s)y_2(t) - y_1(t)y_2(s)).$$

Simplifying,

$$(s^2-1)^2y_1(s) = (s-1)^4, \quad (s^2-1)^2y_2(s) = (s+1)^4.$$

Therefore the integrand is,

$$-\frac{1}{8}(y_2(t)(s-1)^4 - y_1(t)(s+1)^4).$$

Therefore a particular solution is,

$$y_p(t) = \int K(s,t)f(s)ds = -\frac{1}{8}y_2(t)\int (s-1)^4 ds + \frac{1}{8}y_1(t)\int (s+1)^4 ds.$$

Carrying out the antidifferentiations, a particular solution is,

$$-\frac{1}{40}(t-1)^5\left(\frac{t+1}{t-1}\right)^2 + \frac{1}{40}(t+1)^5\left(\frac{t-1}{t+1}\right)^2 = \frac{1}{40}(t^2-1)^2[-(t-1)+(t+1)],$$

i.e.,

$$y_p(t) = \frac{1}{20}(t^2-1)^2.$$

Problem 4 Using the method of undetermined coefficients, find a particular solution of the inhomogeneous linear 2^{nd} order ODE,

$$y'' + 4y' + 5y = 5e^{-2t}\cos(t).$$

Solution: The function $5e^{-2t}\cos(t)$ is the real part of $5e^{-2t}e^{it}$. Therefore a particular solution $y_p(t)$ is the real part of a particular solution $\widetilde{y}_p(t)$ of,

$$\widetilde{y}'' + 4\widetilde{y}' + 5\widetilde{y} = 5e^{-2t}e^{it}.$$

We guess that a particular solution is of the form $\widetilde{y}(t) = e^{(-2+i)t}g(t)$, where $g(t)$ is a polynomial whose coefficients are undetermined. By the exponential shift rule,

$$\begin{gathered}\widetilde{y}'' + 4\widetilde{y}' + 5\widetilde{y} = \\ (D^2+4D+5)e^{(-2+i)t}g(t) = \\ e^{(-2+i)t}((D-2+i)^2 + 4(D-2+i)+5)g(t) = \\ e^{(-2+i)t}(D^2+2iD)g(t)\end{gathered}$$

Therefore $\widetilde{y}(t)$ is a particular solution iff $g(t)$ is a particular solution of the differential equation,

$$(D^2+2iD)g(t) = 5.$$

We guess that $g(t)$ is a linear polynomial with undetermined coefficients and substitute in to determine the coefficients. This gives,

$$g(t) = \frac{5}{2i}t = \frac{-5i}{2}t.$$

Therefore,

$$\widetilde{y}(t) = \frac{-5i}{2} t e^{(-2+i)t} = \frac{5}{2} t e^{-2t}(\sin(t) - i\cos(t)).$$

So a particular solution of the original ODE is,

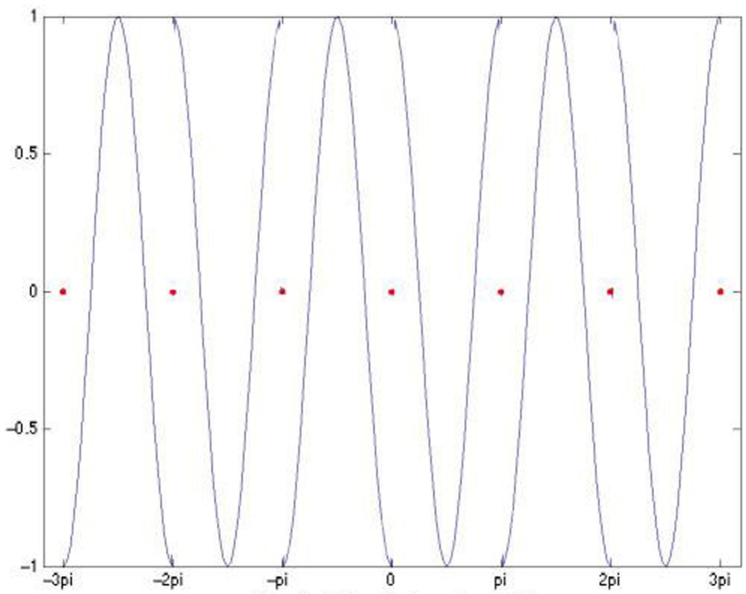
$$y_p(t) = \mathrm{Re}(\widetilde{y}(t)) = \frac{5}{2} t e^{-2t} \sin(t).$$

Problem 5 On the interval $[0, \pi)$, let $f(t) = \cos(2t)$. Denote by $\widetilde{f}(t)$ the odd extension of $f(t)$ as a periodic function of period 2π. Denote by $\mathrm{FSS}[\widetilde{f}]$ the Fourier sine series of $\widetilde{f}(t)$.

(a) Graph $\mathrm{FSS}[\widetilde{f}]$ on the interval $[-3\pi, 3\pi]$. Make special note of all discontinuities and the *actual value* of $\mathrm{FSS}[\widetilde{f}]$ at these points (which does not necessarily agree with the value of $\widetilde{f}$).

Solution: A graph of $\mathrm{FSS}[\widetilde{f}]$ is given below. The points of discontinuity occur for $t = n\pi$, where n is any integer. The value of $\mathrm{FSS}[\widetilde{f}]$ at these points is,

$$\mathrm{FSS}[f](n\pi) = \frac{1}{2}\left(\lim_{t \to n\pi^-} f(t) + \lim_{t \to n\pi^+} f(t)\right) = \frac{1}{2}(1 + (-1)) = 0.$$



Fourier Sine Series of cos(2x)

(b) An orthonormal basis for the odd periodic functions of period 2π is,

$$\phi_n(t) = \frac{1}{\sqrt{\pi}} \sin(nt), \; n = 1, 2, 3, \ldots$$

Compute the Fourier coefficients,

$$a_n = \langle \widetilde{f}, \phi_n \rangle = \int_{-\pi}^{\pi} \widetilde{f}(t)\phi_n(t)dt,$$

and express your answer as a Fourier sine series,

$$\widetilde{f}(t) = \sum_{n=1}^{\infty} a_n \phi_n(t) = \sum_{n=1} \frac{a_n}{\sqrt{\pi}} \sin(nt).$$

Play close attention to $n = 2$.

Solution: By definition,

$$a_n = 2\int_0^{\pi} f(t)\phi_n(t)dt = 2\int_0^{\pi} \frac{1}{\sqrt{\pi}} \cos(2t)\sin(nt)dt.$$

To evaluate this, we use the angle addition formulas,

$$\begin{array}{rcl} \sin(nt+2t) & = & \sin(nt)\cos(2t) \; + \; \sin(2t)\cos(nt), \\ \sin(nt-2t) & = & \sin(nt)\cos(2t) \; - \; \sin(2t)\cos(nt) \end{array}$$

Hence,

$$\cos(2t)\sin(nt) = \frac{1}{2}\left(\sin((n+2)t) + \sin((n-2)t)\right).$$

Substituting this in the integrand gives,

$$a_n = \frac{1}{\sqrt{\pi}} \int_0^{\pi} \sin((n+2)t) + \sin((n-2)t)dt.$$

First of all, for $n = 2$, this gives,

$$a_2 = \frac{1}{\sqrt{\pi}} \int_0^{\pi} \sin(4t) + 1dt = \frac{1}{\sqrt{\pi}} \left(\frac{-1}{4} \cos(4t) + t \right|_0^{\pi} = \sqrt{\pi}.$$

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Continued on next page.

Now suppose that $n \neq 2$. Then both $n+2$ and $n-2$ are invertible. Hence,

$$a_n = \frac{-1}{\sqrt{\pi}}\left(\frac{1}{n+2}\cos((n+2)t) + \frac{1}{n-2}\cos((n-2)t)\right) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{n+2}((-1)^n - 1) + \frac{1}{n-2}((-1)^n - 1)\right).$$

If n is even, then $(-1)^n = 1$ and $a_n = 0$. If n is odd, then $(-1)^n - 1 = -2$ and,

$$a_n = \frac{2}{\sqrt{\pi}}\left(\frac{1}{n+2} + \frac{1}{n-2}\right) = \frac{2}{\sqrt{\pi}}\frac{2n}{n^2-4} = \frac{4n}{\sqrt{\pi}(n^2-4)}.$$

Therefore the Fourier sine series of $\widetilde{f}(t)$ is,

$$\mathrm{FSS}[\widetilde{f}] = \sin(2t) + \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{2m+1}{(2m+3)(2m-1)}\sin((2m+1)t).$$

Problem 6 On the interval $[-\pi, \pi)$, let $f(t)$ be the square-wave function,

$$f(t) = \begin{cases} 1, & 0 \leq t < \pi, \\ -1, & -\pi \leq t < 0 \end{cases}$$

Let $\widetilde{f}(t)$ be the extension of $f(t)$ to a periodic function of period 2π. An orthonormal basis for the periodic functions of period 2π is,

$$\phi_n(t) = \frac{1}{\sqrt{2\pi}}e^{int}, \quad n \in \mathbb{Z}.$$

(a) Compute the Fourier coefficients,

$$a_n(\widetilde{f}) = \langle \widetilde{f}, \phi_n \rangle = \int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}f(t)e^{-int}dt.$$

Solution: The Fourier coefficient is,

$$a_n = \int_{-\pi}^{0}\frac{1}{\sqrt{2\pi}}(-1)e^{-int}dt + \int_{0}^{\pi}\frac{1}{\sqrt{2\pi}}(1)e^{-int}dt.$$

In the first integral, make the substitution $u = -t$ to get,

$$a_n = \frac{-1}{\sqrt{2\pi}}\int_0^{\pi}(e^{int} - e^{-int})dt = \frac{-2i}{\sqrt{2\pi}}\int_0^{\pi}\sin(nt)dt.$$

If $n = 0$ then $a_0 = 0$. If $n \neq 0$ then,

$$a_n = \frac{2i}{\sqrt{2\pi}}\left(\frac{1}{n}\cos(nt)\right|_0^{\pi} = \frac{2i}{\sqrt{2\pi}n}((-1)^n - 1).$$

If n is even, then $a_n = 0$. If n is odd, then,

$$a_n = \frac{-4i}{\sqrt{2\pi}n}.$$

(b) Denote by $y(t)$ the periodic function of period 2π that solves the ODE,

$$y'' + 4y' + 5y = \widetilde{f}(t).$$

Using (a), compute the Fourier coefficients,

$$a_n(y) = \langle y, \phi_n \rangle.$$

Solution: Of course $a_n(y') = ina_n(y)$ and $a_n(y'') = -n^2 a_n(y)$. Taking Fourier coefficients of each side of the equation gives,

$$(-n^2 + 4in + 5)a_n(y) = a_n(\widetilde{f}).$$

For every integer n, $-n^2 + 4in + 5$ is nonzero. Therefore $a_n(y)$ is zero unless n is odd. In this case,

$$(-n^2 + 4in + 5)a_n(y) = \frac{-4i}{\sqrt{2\pi}n}.$$

Multiplying each side by the complex conjugate $((5 - n^2) - 4in)$ and solving for $a_n(y)$ gives,

$$a_n(y) = \begin{cases} i|a_n|e^{-i\psi_n}, & n \text{ a positive odd integer,} \\ -i|a_n|e^{i\psi_n}, & n \text{ a negative odd integer} \end{cases}$$

where,

$$|a_n| = \frac{4}{\sqrt{2\pi}|n|}\frac{1}{\sqrt{(n^2+3)^2+16}},$$

and,

$$\tan(\psi_n) = \frac{4|n|}{n^2 - 5}.$$

This last equation only determines ψ_n up to a multiple of π. For $n = 1$, $\psi_1 = \frac{3\pi}{4}$. For all other odd values of n, $0 < \psi_n < \frac{\pi}{2}$.

(c) Write down just the terms in the Fourier series of $y(t)$ coming from $n = -1$ and $n = 1$. Express in terms of sine and cosine.

Solution: First of all, for each positive odd integer n, observe that the terms coming from n and $-n$ are,

$$\frac{1}{\sqrt{2\pi}}|a_n|ie^{-i\psi_n}e^{inx} - \frac{1}{\sqrt{2\pi}}|a_n|ie^{i\psi_n}e^{-inx} = \frac{i|a_n|}{\sqrt{2\pi}}(e^{i(nx-\psi_n)} - e^{-i(nx-\psi_n)}) = \frac{-2|a_n|}{\sqrt{2\pi}}\sin(nx - \psi_n).$$

For $n = 1$, $|a_1| = 1/(2\sqrt{\pi})$ and $\psi_n = 3\pi/4$. Therefore the terms coming from $n = \pm 1$ are,

$$\frac{1}{\sqrt{2\pi}}\sin(x - \pi/4).$$

Problem 7 Consider the IVP,

$$\begin{cases} y'' - 4y' + 5y = 3e^{2t}\sin(t), \\ y(0) = 1, \\ y'(0) = 0 \end{cases}$$

Denote by $Y(s)$ the Laplace transform,

$$\mathcal{L}[y(t)] = \int_0^\infty e^{-st}y(t)dt.$$

(a) Compute the Laplace transform of the IVP and use this to find an equation that $Y(s)$ satisfies.

Solution: The Laplace transform of the left-hand-side is,

$$\mathcal{L}[y''-4y'+5y] = (s^2Y-s)-4(sY-1)+5Y = (s^2-4s+5)Y-(s-4) = ((s-2)^2+1)Y(s)-(s-2)-(-2).$$

The Laplace transform of the right-hand-side is,

$$\mathcal{L}[3e^{2t}\sin(t)] = 3\mathcal{L}[\sin(t)](s-2) = \frac{3}{(s-2)^2+1}.$$

Therefore $Y(s)$ satisfies the equation,

$$((s-2)^2+1)Y(s) = (s-2) + (-2) + \frac{3}{(s-2)^2+1}.$$

(b) Solve the equation for $Y(s)$. No partial fractions are needed.

Now suppose that $n \neq 2$. Then both $n+2$ and $n-2$ are invertible. Hence,

$$a_n = \frac{-1}{\sqrt{\pi}}\left(\frac{1}{n+2}\cos((n+2)t) + \frac{1}{n-2}\cos((n-2)t)\right) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{n+2}((-1)^n - 1) + \frac{1}{n-2}((-1)^n - 1)\right).$$

If n is even, then $(-1)^n = 1$ and $a_n = 0$. If n is odd, then $(-1)^n - 1 = -2$ and,

$$a_n = \frac{2}{\sqrt{\pi}}\left(\frac{1}{n+2} + \frac{1}{n-2}\right) = \frac{2}{\sqrt{\pi}}\frac{2n}{n^2-4} = \frac{4n}{\sqrt{\pi}(n^2-4)}.$$

Therefore the Fourier sine series of $\widetilde{f}(t)$ is,

$$\mathrm{FSS}[\widetilde{f}] = \sin(2t) + \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{2m+1}{(2m+3)(2m-1)}\sin((2m+1)t).$$

Problem 6 On the interval $[-\pi, \pi)$, let $f(t)$ be the square-wave function,

$$f(t) = \begin{cases} 1, & 0 \leq t < \pi, \\ -1, & -\pi \leq t < 0 \end{cases}$$

Let $\widetilde{f}(t)$ be the extension of $f(t)$ to a periodic function of period 2π. An orthonormal basis for the periodic functions of period 2π is,

$$\phi_n(t) = \frac{1}{\sqrt{2\pi}}e^{int}, \quad n \in \mathbb{Z}.$$

(a) Compute the Fourier coefficients,

$$a_n(\widetilde{f}) = \langle \widetilde{f}, \phi_n \rangle = \int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}f(t)e^{-int}dt.$$

Solution: The Fourier coefficient is,

$$a_n = \int_{-\pi}^{0}\frac{1}{\sqrt{2\pi}}(-1)e^{-int}dt + \int_{0}^{\pi}\frac{1}{\sqrt{2\pi}}(1)e^{-int}dt.$$

In the first integral, make the substitution $u = -t$ to get,

$$a_n = \frac{-1}{\sqrt{2\pi}}\int_0^{\pi}(e^{int} - e^{-int})dt = \frac{-2i}{\sqrt{2\pi}}\int_0^{\pi}\sin(nt)dt.$$

If $n = 0$ then $a_0 = 0$. If $n \neq 0$ then,

$$a_n = \frac{2i}{\sqrt{2\pi}}\left(\frac{1}{n}\cos(nt)\right|_0^{\pi} = \frac{2i}{\sqrt{2\pi}n}((-1)^n - 1).$$

If n is even, then $a_n = 0$. If n is odd, then,

$$a_n = \frac{-4i}{\sqrt{2\pi}n}.$$

(b) Denote by $y(t)$ the periodic function of period 2π that solves the ODE,

$$y'' + 4y' + 5y = \widetilde{f}(t).$$

Using (a), compute the Fourier coefficients,

$$a_n(y) = \langle y, \phi_n \rangle.$$

Solution: Of course $a_n(y') = ina_n(y)$ and $a_n(y'') = -n^2a_n(y)$. Taking Fourier coefficients of each side of the equation gives,

$$(-n^2 + 4in + 5)a_n(y) = a_n(\widetilde{f}).$$

Hence a basis for the solution space is,

$$y_1(t) = e^{2t} \begin{bmatrix} \cos(t) \\ 2\cos(t) - \sin(t) \end{bmatrix}, \quad y_2(t) = e^{2t} \begin{bmatrix} \sin(t) \\ \cos(t) + 2\sin(t) \end{bmatrix}.$$

Therefore the solution matrix $X(t)$ is,

$$X(t) = e^{2t} \begin{bmatrix} \cos(t) & \sin(t) \\ 2\cos(t) - \sin(t) & \cos(t) + 2\sin(t) \end{bmatrix}.$$

(d) Compute the exponential matrix,

$$\exp(tA) = X(t)X(0)^{-1}.$$

Solution: The matrix $X(0)$ is,

$$X(0) = e^0 \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Therefore the inverse matrix is,

$$X(0)^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Therefore the exponential matrix is,

$$\exp(tA) = X(t)X(0)^{-1} =$$

$$e^{2t} \begin{bmatrix} \cos(t) & \sin(t) \\ 2\cos(t) - \sin(t) & \cos(t) + 2\sin(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} =$$

$$e^{2t} \begin{bmatrix} \cos(t) - 2\sin(t) & \sin(t) \\ -5\sin(t) & \cos(t) + 2\sin(t) \end{bmatrix}.$$

(e) Denote by $\mathbf{f}(t)$ the vector-valued function,

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ 3e^{2t}\sin(t) \end{bmatrix}.$$

Denote by $\mathbf{x}_0$ the column vector,

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For the following IVP write down the solution in terms of the matrix exponential.

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

Compute the entries of the constant term vector and the integrand column vector, but do not evaluate the integrals.

Solution: The solution of the IVP is,

$$\mathbf{x}(t) = \exp(tA)\mathbf{x}_0 + \exp(tA) \int_0^t \exp(-uA)\mathbf{f}(u)du.$$

First of all,

$$\exp(tA)\mathbf{x}_0 = e^{2t} \begin{bmatrix} \cos(t) - 2\sin(t) & \sin(t) \\ -5\sin(t) & \cos(t) + 2\sin(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} \cos(t) - 2\sin(t) \\ -5\sin(t) \end{bmatrix}.$$

Next,

$$\exp(-uA)\mathbf{f}(u) = e^{-2u}\begin{bmatrix} \cos(u)+2\sin(u) & -\sin(u) \\ 5\sin(u) & \cos(u)-2\sin(u) \end{bmatrix}\begin{bmatrix} 0 \\ 3e^{2u}\sin(u) \end{bmatrix}.$$

This equals,

$$\begin{bmatrix} -3\sin^2(u) \\ 3\sin(u)\cos(u) - 6\sin^2(u) \end{bmatrix}.$$

So the integral is,

$$\int_0^t \exp(-uA)\mathbf{f}(u)du = \int_0^t \begin{bmatrix} -3\sin^2(u) \\ 3\sin(u)\cos(u) - 6\sin^2(u) \end{bmatrix} du.$$

Finally, without evaluating the integral, the solution is,

$$\mathbf{x}(t) = e^{2t}\begin{bmatrix} \cos(t) - 2\sin(t) \\ -5\sin(t) \end{bmatrix} +$$

$$e^{2t}\begin{bmatrix} \cos(t)-2\sin(t) & \sin(t) \\ -5\sin(t) & \cos(t)+2\sin(t) \end{bmatrix}\int_0^t \begin{bmatrix} -3\sin^2(u) \\ 3\sin(u)\cos(u) - 6\sin^2(u) \end{bmatrix} du.$$

Problem 9 Consider the following nonlinear, autonomous planar system,

$$\begin{cases} x' &= (x+y)(x-1) \\ y' &= (x-y)(x+1) \end{cases}$$

(a) Find all equilibrium points.

Solution: The equilibrium points are the simultaneous solutions of,

$$\begin{cases} (x+y)(x-1) &= 0, \\ (x-y)(x+1) &= 0 \end{cases}$$

These are $p_1 = (0,0)$, $p_2 = (1,1)$ and $p_3 = (-1,1)$.

(b) Determine the linearization at each equilibrium point.

Solution: The Jacobian of the system is,

$$J(x,y) = \left[\frac{\partial F_i}{\partial x_j}\right] = \begin{bmatrix} 2x+y-1 & x-1 \\ 2x-y+1 & -x-1 \end{bmatrix}.$$

Therefore the linearization at p_i is $\mathbf{x}' = A_i\mathbf{x}$, where,

$$A_1 = J(0,0) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix},$$

where,

$$A_2 = J(1,1) = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix},$$

and where,

$$A_3 = J(-1,1) = \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix}.$$

(c) For each linearization, determine the eigenvalues. If the eigenvalues are complex conjugates, determine the rotation (clockwise in/out, counterclockwise in/out). If the eigenvalues are real, determine roughly the eigenvectors and the type of the local phase portrait.

Solution: For p_1, Trace$(A_1) = -2$ and $\det(A_1) = 2$. So $p_{A_1}(\lambda) = \lambda^2 + 2\lambda + 2 = (\lambda+1)^2 + 1$. Therefore the eigenvalues are $\lambda_\pm = -1 \pm i$. So the type is a stable spiral. Checking at a few representative points, the rotation is counterclockwise in.

For p_2, Trace$(A_2) = 0$ and $\det(A_2) = -4$. So $p_{A_2}(\lambda) = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$. Therefore the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$. So the type is a saddle. An eigenvector for $\lambda_1 = 2$ is,

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

An eigenvector for $\lambda_2 = -2$ is,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For p_3, Trace$(A_2) = -2$ and $\det(A_3) = -4$. So $p_{A_3}(\lambda) = \lambda^2 + 2\lambda - 4 = (\lambda + 1)^2 - 5$. Therefore the eigenvalues are $\lambda_3 = \sqrt{5} - 1 \approx 1$ and $\lambda_4 = -\sqrt{5} - 1 \approx -3$. So the type is a saddle. An eigenvector for $\lambda_3 = \sqrt{5} - 1$ is,

$$\mathbf{v}_3 = \begin{bmatrix} -\sqrt{5} + 1 \\ 2 \end{bmatrix} \approx \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

And an eigenvector for $\lambda_4 = -\sqrt{5} - 1$ is,

$$\mathbf{v}_4 = \begin{bmatrix} \sqrt{5} + 1 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

(d) Using a dashed line, sketch the x-nullcline and y-nullcline. Draw a few representative arrows indicating the direction of the orbits on the nullcline on each side of each equilibrium point.

Solution: The x-nullcline consists of the vertical line $x = 1$ and the "antidiagonal" line $y = -x$. On the line $x = 1$, y' is negative for $y > 1$ and y' is positive for $y < 1$. On the line $y = -x$, y' is positive for $x > 0$, y' is negative for $-1 < x < 0$, and y' is positive for $x < -1$. Notice in particular that the line $x = 1$ is a union of orbits.

The y-nullcline consists ofthe vertical line $x = -1$ and the diagonal line $y = x$. On the line $x = -1$, x' is negative for $y > 1$ and x' is positive for $y < 1$. On the line $y = x$, x' is positive for $x > 1$, x' is negative for $0 < x < 1$, and x' is positive for $x < 0$.

It is also worth noting that on the horizontal line $y = 1$, the vector field is $(x^2 - 1, x^2 - 1)$. In particular, the direction is the same as $(-1, -1)$ on the line segment $y = 1$, $|x| < 1$. Therefore any orbit that enters the strip bounded by the following line segments is in the basin of attraction of p_1,

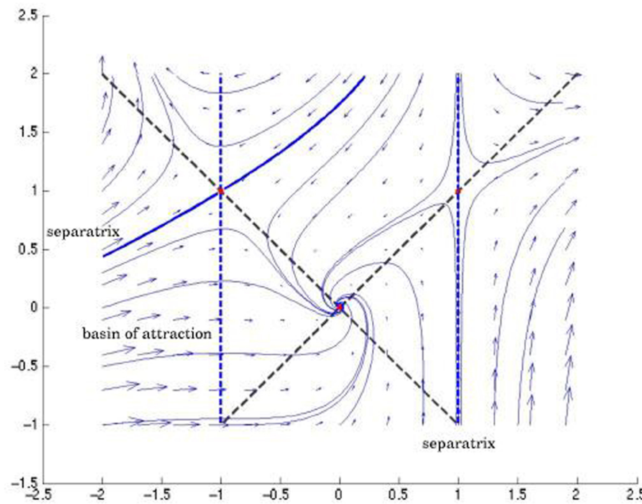
$$\begin{cases} x = -1 \\ y < 1 \end{cases}$$

$$\begin{cases} y = 1 \\ -1 < x < 1 \end{cases}$$

$$\begin{cases} x = 1 \\ y < 1 \end{cases}$$

(e) Sketch the phase portrait. Use bold lines to indicate each (rough) separatrix. Label the basins of attraction (with the numbers 1, 2, 3, etc. placed at some point in the basin of attraction). Your sketch should just be a rough sketch, but it should be qualitatively correct.

Solution: The orbital portrait is on the webpage. There is only a single basin of attraction – the one associated to p_1. On the right it is bounded by the vertical line $x = 1$. On the left it is bounded by the curve made up of two orbits whose forward limit set is p_3, each of which is asymptotic as $t \to \infty$ to $p_3 + C\mathbf{v}_4 e^{\lambda_4 t}$, for $C > 0$ and for $C < 0$. The left separatrix is solid blue in the image given below.


separatrix
basin of attraction
separatrix