18.034 PROBLEM SET 3

Due date: Friday, March 5 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Each of the following problems is from the textbook. The point value of the problem is next to the problem. In Problem 36 from p. 175, please give an accurate plot. If you choose to do this by hand, that is fine, but plot the whole interval $[-2, 0]$ with a step size of 0.1. If you choose to use Matlab, there will be a handout online giving step-by-step instructions how to plot functions.

(1)(5 points) p. 129, Problem 4

Solution: First there is a preliminary estimate that is useful. Let ϵ be a real number such that $0 < \epsilon < 1$. Define $\eta = 1 - \epsilon$. If $|u| \leq \eta$, then $|1 + u|^2 \geq \epsilon^2$ and $1/|1 + u|^2 \leq 1/\epsilon^2$. Applying the Mean Value Theorem to $f(u) = \log(1 + u) - u$ and then to $f'(u) = \frac{1}{1+u} - 1$ yields,

$$-\frac{|u|^2}{\epsilon^2} \leq \log(1 + u) - u \leq \frac{|u|^2}{\epsilon^2}, \text{ if } |u| < \eta.$$

Because the exponential function is monotone, this gives the following estimate,

$$1 - \exp(n|u|^2/\epsilon^2) \leq \exp(0) - \exp(n \log(1 + u) - nu) \leq 1 - \exp(-n|u|^2/\epsilon^2), \text{ if } |u| < \eta.$$

Simplifying this expression and multiplying both sides by e^{nu} gives the estimate,

$$e^{nu}(1 - e^{n|u|^2/\epsilon^2}) \leq (1 + u)^n - e^{nu} \leq e^{nu}(1 - e^{-n|u|^2/\epsilon^2}), \text{ if } |u| < \eta.$$

Now consider the Euler's Method approximation for the linear, first-order, constant coefficient IVP,

$$\begin{cases} y' = ay, \\ y(t_0) = y_0 \end{cases}$$

with step size h. Consider only h such that $|a|h < \eta$. Denote $y_n = y(t_0 + nh)$ for n any integer. Euler's Method gives the following recursion relation,

$$y_{n+1} = y_n + hay_n = (1 + ah)y_n.$$

The solution of this recursion relation is,

$$y_n = (1 + ah)^n y_0,$$

and the piecewise linear Euler Solution is,

$$y_h(t) = (1 + ah)^n y_0(1 + ah(t - nh)),$$

for $t \in [t_0 + nh, t_0 + (n + 1)h]$. The true solution is readily computed as

$$\widetilde{y}(t) = y_0 e^{a(t - t_0)}.$$

Let t be any real number, and let n be the largest integer such that $t_0 + nh \leq t$. Then,

$$|y_h(t) - \widetilde{y}(t)| \leq |y_h(t) - y_h(t_n)| + |y_h(t_n) - \widetilde{y}(t_n)| + |\widetilde{y}(t_n) - \widetilde{y}(t)|$$
$$\leq \left(|y_0||a|h(1 + ah)^n\right) + \left(|y_0||(1 + ah)^n - e^{nah}|\right) + \left(|y_0||e^{a(t_n-t)} - 1||e^{a(t-t_0)}|\right).$$

The first and third terms clearly become arbitrarily small as $h \to 0$ (so that also $t_n - t \to 0$). For the middle term, setting $u = ah$, the preliminary estimate gives,

$$|y_0|e^{nah}(1 - e^{|n||a|^2 h^2/\epsilon^2}) \leq |y_0|(1 + ah)^n - e^{nah} \leq |y_0|e^{nah}(1 - e^{-|n||a|^2 h^2/\epsilon^2}).$$

As $h \to 0$, the quantity nh coverges to $t - t_0$. But the quantity nh^2 converges to 0. Therefore, as $h \to 0$, both bounds converge to 0. By the Squeezing Lemma, the middle term converges to 0 as $h \to 0$. Therefore, as $h \to 0$, the Euler Solutions $y_h(t)$ converge to the true solution $\widetilde{y}(t)$.

(2)(5 points) p. 129, Problem 5

Solution: Let $[a, b]$ be an interval with $b > a$. Let $n > 0$ be an integer. For $k = 0, \ldots 2n$, define $t_k = a + \frac{b-a}{2n}k$. The subintervals $[t_k, t_{k+1}]$ give a partition of $[a, b]$ into $2n$ equally-spaced subintervals of length $\frac{b-a}{2n}$. The Simpson's Rule approximation for this partition is,

$$\int_a^b f(t)dt \approx \frac{b-a}{6n}\left[f(t_0) + \sum_{k=1}^{n-1}(4f(t_{2k-1}) + 2f(t_{2k})) + 4f(t_{2n-1}) + f(t_{2n})\right] =$$
$$\frac{b-a}{6n}\left[f(t_0) + 4f(t_1) + 2f(t_2) + 4f(t_3) + 2f(t_4) + \cdots + 2f(t_{2n-2}) + 4f(t_{2n-1}) + f(t_{2n})\right].$$

Observe that for each $k = 0, \ldots, n-1$, Simpson's Rule applied to the subinterval $[t_{2k}, t_{2k+2}]$ broken into 2 equally spaced subintervals gives,

$$\int_{t_{2k}}^{t_{2k+2}} f(t)dt \approx \frac{b-a}{6n}\left[f(t_{2k}) + 4f(t_{2k+1}) + f(t_{2k+2})\right].$$

Forming the sum of each of these approximations for $k = 0, \ldots, n-1$ gives the usual Simpson's rule.

Apply RK4 to the ODE,

$$\begin{cases} y' = f(t), \\ y(t_0) = y_0 \end{cases}$$

with $[a, b]$ broken into n equal subintervals of length $\frac{b-a}{n}$, i.e.

$$[t_0, t_2], [t_2, t_4], \ldots, [t_{2k}, t_{2k+2}], \ldots, [t_{2n-2}, t_{2n}].$$

The claim is the RK4 approximation for $y(b)$ equals the Simpson's Rule approximation. By the last paragraph, the Simpson's Rule approximation is the sum of the Simpson's Rule approximation for each subinterval $[t_{2k}, t_{2k+2}]$. So it suffices to prove for each $k = 0, \ldots, n-1$, that the RK4 approximation for $y_{k+1} - y_k$ equals the Simpson's Rule approximation for $[t_{2k}, t_{2k+2}]$ broken into 2 equal subintervals.

By definition, the RK4 approximation gives,

$$y_{n+1} - y_n = \frac{b-a}{6n}(f(t_{2n}) + 2f(t_{2n+1}) + 2f(t_{2n+1}) + f(t_{2n+2})).$$

Simplifying, this is precisely the Simpson's Rule approximation,

$$\int_{t_{2n}}^{t_{2n+2}} f(t)dt \approx \frac{b-a}{6n}(f(t_{2n}) + 4f(t_{2n+1}) + f(t_{2n+2})).$$

(3)(5 points) p. 168, Problem 16

Solution: One solution is clearly the constant function $y(t) = 0$. By the uniqueness theorem, Theorem 3.1.1, this is the unique solution.

(4)(5 points) p. 175, Problem 4

Solution: The characteristic polynomial is $r^2 + r - 2$. Every root that is a rational number is $\pm$ a fraction whose numerator is a divisor of the constant coefficient, and whose denominator is a divisor of the leading coefficient. In this case, ± 1 or ± 2. Plugging in, $r = 1$ and $r = -2$ are roots (this also follows from the quadratic formula). Therefore, by Theorem 3.2.1, the general solution is,

$$C_1 e^{-2t} + C_2 e^t.$$

(5)(5 points) p. 175, Problem 6

Solution: The characteristic polynomial of the normal form is $r^2 + 6r^9$. By the same method as above, -3 is a repeated root of the characteristic polynomial. Therefore, by Theorem 3.2.1, the general solution is,

$$C_1 e^{-3t} + C_2 t e^{-3t}.$$

(6)(5 points) p. 175, Problem 36

Solution: The characteristic polynomial is $r^2 - 4r + 4$. By the same method as above, 2 is a repeated root. Therefore, by Theorem 3.2.1, the general solution is,

$$C_1 e^{2t} + C_2 t e^{2t}.$$

It is equivalent, but slightly more useful, to write this as,
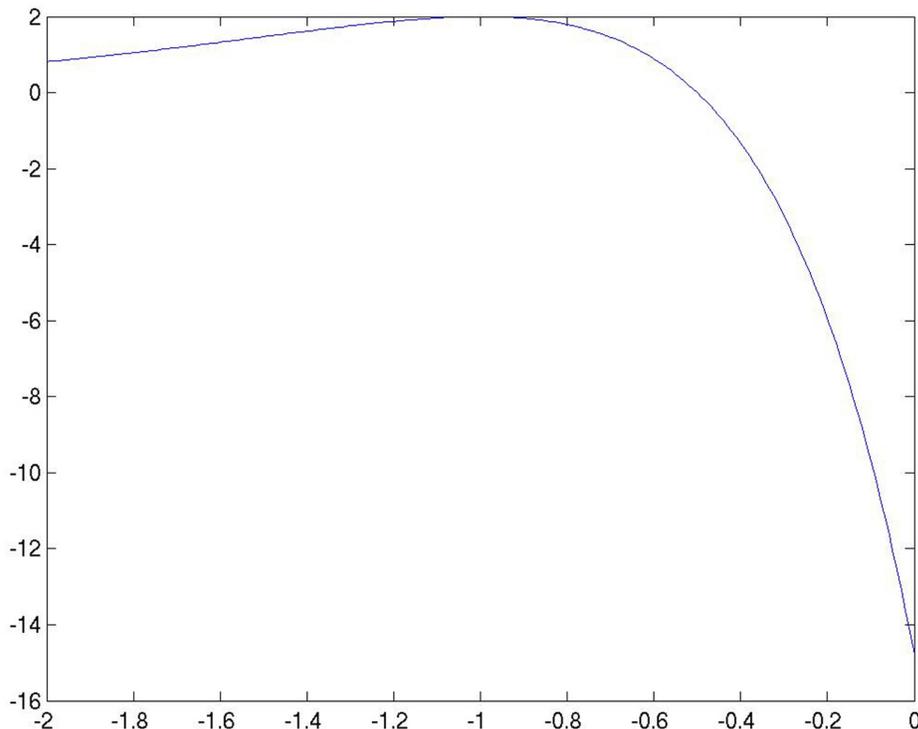
$$D_1 e^{2(t+1)} + D_2 (t+1) e^{2(t+1)}.$$

The initial conditions give,

$$\begin{cases} D_1 & = & 2 \\ 2D_1 & + & D_2 & = & 0 \end{cases}$$

So $D_1 = 2$ and $D_2 = -4$. The solution of the IVP is,

$$y(t) = 2e^{2(t+1)} - 4(t+1)e^{2(t+1)}$$
$$= -2e^2 e^{2t} - 4e^2 t e^{2t}.$$

The graph of the solution is given below.

(7)(5 points) p. 175, Problem 44

Solution: Of course Abel's Theorem proves this. Most likely the exercise is asking for the formula for the Wronskian of each solution pair. In the first case,

$$\begin{aligned} y_1(t) &= e^{r_1 t}, & y_2(t) &= e^{r_2 t} \\ y_1'(t) &= r_1 e^{r_1 t}, & y_2'(t) &= r_2 e^{r_2 t}. \end{aligned}$$

So the Wronskian is,

$$W[y_1, y_2](t) = (r_2 - r_1)e^{(r_1+r_2)t}.$$

Because $r_2 \neq r_1$, the coefficient is nonzero. And the exponential function is always nonzero. So $W[y_1, y_2](t)$ is always nonzero.

In the second case,

$$\begin{aligned} y_1(t) &= e^{r_1 t}, & y_2(t) &= t e^{r_1 t} \\ y_1'(t) &= r_1 e^{r_1 t}, & y_2'(t) &= (r_1 t + 1)e^{r_1 t}. \end{aligned}$$

So the Wronskian is,

$$W[y_1, y_2](t) = (r_1 t + 1)e^{2r_1 t} - r_1 t e^{2r_1 t} = e^{2r_1 t}.$$

Because the exponential is always nonzero, $W[y_1, y_2](t)$ is always nonzero.

In the final case,

$$\begin{aligned} y_1(t) &= e^{\alpha t}\cos(\beta t), & y_2(t) &= e^{\alpha t}\sin(\beta t) \\ y_1'(t) &= \alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t), & y_2'(t) &= \alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\cos(\beta t) \end{aligned}$$

So the Wronskian is,

$$\begin{aligned} W[y_1, y_2](t) &= e^{2\alpha t}(\alpha\sin(\beta t)\cos(\beta t) + \beta\cos(\beta t)^2) - e^{2\alpha t}(\alpha\cos(\beta t)\sin(\beta t) - \beta\sin(\beta t)^2) \\ &= \beta e^{2\alpha t}(\cos(\beta t)^2 + \sin(\beta t)^2) = \beta e^{2\alpha t}. \end{aligned}$$

By hypothesis, $\beta \neq 0$. And the exponential is always nonzero. Therefore $W[y_1, y_2](t)$ is always nonzero.

(8)(5 points) p. 175, Problem 46

Solution: The characteristic polynomial is $r^2 - r - 2$. The roots are -1 and 2. Therefore, by Theroem 3.2.1, the general solution is,

$$C_1 e^{-t} + C_2 e^{2t}.$$

Therefore, two solutions are,

$$\begin{aligned} y_1(t) &= e^{-t} + e^{2t}, \\ y_2(t) &= e^{-t} - e^{2t}. \end{aligned}$$

The derivatives are,

$$\begin{aligned} y_1'(t) &= -e^{-t} + 2e^{2t}, \\ y_2'(t) &= -e^{-t} - 2e^{2t}. \end{aligned}$$

Therefore the Wronskian is,

$$W[y_1, y_2](t) = -(e^{-2t} + 3e^t + 2e^{4t}) - (-e^{-2t} + 3e^t - 2e^{4t}) = -6e^t.$$

Because the exponential is always nonzero, $W[y_1, y_2](t)$ is always nonzero. Therefore (y_1, y_2) is a basic solution pair (one of infinitely many!).

(9)(10 points) p. 176, Problem 48

Solution, (a): By Theorem 3.2.1, the general solution is,

$$C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t).$$

Define $A = \sqrt{C_1^2 + C_2^2}$. Define ϕ to be the unique angle, $-\pi < \phi \le \pi$ such that $\tan(\phi) = C_2/C_1$. Then,

$$\begin{aligned} C_1 &= A\cos(\phi), \\ C_2 &= A\sin(\phi). \end{aligned}$$

By the angle addition formulas,

$$\begin{aligned} y(t) &= C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t) \\ &= A e^{\alpha t} (\cos(\beta t) \cos(\phi) + \sin(\beta t) \sin(\phi)) \\ &= A e^{\alpha t} \cos(\beta t - \phi). \end{aligned}$$

Of course every nonzero solution occurs for a unique $A > 0$ and a unique $-\pi < \phi \le \pi$. On the other hand, it is straightforward to check for every real number A (positive, negative or zero) and every angle ϕ, $A e^{\alpha t} \cos(\beta t - \phi)$ is a solution.

(b). Define the *period* to be $T = \frac{2\pi}{\beta}$. Then for the general solution $y(t)$ above and for every integer n,

$$y(t + nT) = A e^{\alpha(t+nT)} \cos(\beta t + 2n\pi - \phi) = e^{\alpha nT} y(t).$$

Similarly, for every integer n,

$$y(t + (n+1/2)T) = A e^{\alpha(t+(n+1/2)T)} \cos(\beta t + 2n\pi + \pi - \phi) = -e^{\alpha(n+1/2)T} y(t).$$

Therefore if $y_1(t)$ and $y_2(t)$ are two particular solutions such that $y_1(t_0) = y_2(t_0)$, then for every integer n,

$$y_1(t_0 + nT) = y_2(t_0 + nT), \quad y_1(t_0 + (n+1/2)T) = y_2(t_0 + (n+1/2)T).$$

Therefore the nodes are equally spaced at intervals of $\frac{1}{2}T = \frac{\pi}{\beta}$.