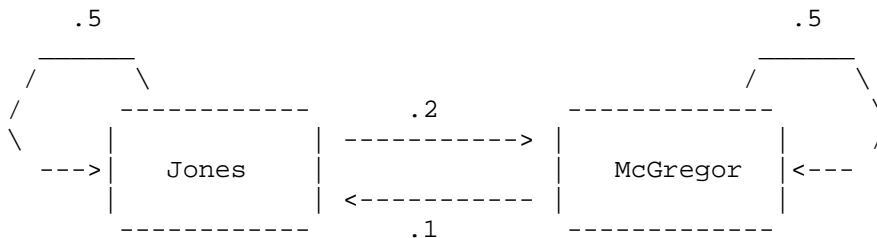


Linear first order systems: Introduction

1. Elimination
2. Matrices
3. Anti-elimination: companion matrices
4. Shakespeare

[1] There are two fields in which rabbits are breeding like rabbits, Farmer Jones's field contains  $x(t)$  rabbits, Farmer McGregor's field contains  $y(t)$  rabbits. Breeding like rabbits in this case means that Jones's show a net growth rate of .7 rabbits per rabbit per month, and McGregor's show .6. They can also hop over the hedge between the fields, and since the grass is greener in Farmer McGregor's field, Jones's rabbits jump over at the rate of .2 per month, MacGregor's at the rate of .1 per month.

Flow diagram:



So the equations are

$$x' = .5x - .2x + .1y = .3x + .1y \quad (1)$$

$$y' = .5y - .1y + .2x = .2x + .4y \quad (2)$$

This is a linear SYSTEM of equations, homogeneous, and it seems to be impossible to solve, since you need to know  $y$  to solve for  $x$  and you need to know  $x$  to solve for  $y$ .

We *can* solve, though: use (1) to express  $y$  in terms of  $x$  :

$$y = 10 x' - 3 x$$

and then plug this into (2):

$$10 x'' - 3 x' = .2 x + 4x' - 1.2 x$$

$$10 x'' - 7 x' + x = 0$$

or

$$x'' - .7 x' + .1 x = 0$$

This is a SECOND ORDER ODE , which we can solve:

$$s^2 - .7s + .1 = 0$$

has roots  $r_1 = .5$  and  $r_2 = .2$  .

(Remember,  $(s - r_1)(s - r_2) = s^2 - (r_1 + r_2) s + r_1 r_2$  .)

So we get two basic solutions,

$$\begin{aligned}x_1 &= e^{.5 t} \\x_2 &= e^{.2 t}\end{aligned}$$

Each gives a corresponding solution for  $y$  , using  $y = 10 x' - 3 x$  :

$$\begin{aligned}y_1 &= 2 e^{.5 t} \\y_2 &= - e^{.1 t}\end{aligned}$$

Well, some of these solutions are not biologically significant.

We can package this information in a vector:  $u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

This vector traces a curve in the plane, as time increases.

For example, the solution  $u_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  traces a ray that passes

through  $(1,2)$  at  $t = 0$  and move off towards infinity like  $e^{.5t}$  .

[2] Analysis of homogeneous linear equations by matrices.

$$\begin{aligned}x' &= ax + by \\y' &= cx + dy\end{aligned} \quad (*)$$

A  $2 \times 2$  matrix is an array of numbers (enclosed by brackets)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

In these notes I will use Matlab notation and write this array as

$$[ a \ b ; \ c \ d ]$$

There is another matrix in sight, the "column vector"

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

or  $[ x ; y ]$  with entries  $x$  and  $y$  .

Matrix multiplication is set up so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

or  $[ a \ b ; \ c \ d ] [ x ; y ] = [ ax+by ; cx+dy ]$

You run along a row of the first matrix and a column of the second, multiplying and adding up. Another way to look at matrix multiplication comes from rewriting the right hand side as

$$\begin{vmatrix} ax + by \\ cx + dy \end{vmatrix} = x \begin{vmatrix} a \\ c \end{vmatrix} + y \begin{vmatrix} b \\ d \end{vmatrix}$$

or  $[ ax+by ; cx + dy ] = x [ a ; c ] + y [ b ; d ]$

The ODE (\*) can thus be written as

$$[ x' ; y' ] = [ a \ b ; c \ d ] [ x ; y ]$$

If we write  $u$  for the column vector  $[ x ; y ]$  then  $u' = [ x' ; y' ]$ , and

$$u' = Au$$

This compact expression is exactly equivalent to (\*).

In the example we just gave, there are two independent basic solutions:

$$\begin{aligned} u_1 &= e^{.5t} [ 1 ; 2 ] \\ u_2 &= e^{.2t} [ 1 ; -1 ] \end{aligned}$$

and the general solution is a linear combination of these two.

[3] Companion matrices.

Any second order linear equation occurs in this way, and in fact the first order system gives an important way to visualize the second order equation.

Suppose we have a second order homogeneous linear equation, say

$$x'' + b x' + k x = 0 \quad (**)$$

We can derive a first order linear system from this, by the trick of defining

$$y = x'$$

so then

$$y' = x'' = - b x' - k x = - kx - b y$$

The corresponding matrix is  $A = [ 0 , 1 ; -k , -b ]$ . This is the "companion matrix" of the equation (\*\*).

The solutions to  $u' = A u$  are of the form  $u = [ x ; x' ]$ ; it records both the solution to (\*\*) and its derivative.

For example,

$$x'' - (1/2)x' + (17/16)x = 0$$

The companion matrix is  $[ 0 , 1 ; -17/16 , 1/2 ]$ .

What do solutions of this one look like? The characteristic polynomial of the second order equation is

$$p(s) = s^2 - (1/2)s + (17/16) = (s - (1/4))^2 + 1$$

so a root  $r$  satisfies  $(r - (1/4))^2 = -1$  or  $r = (1/4) \pm i$   
 The general solution is thus

$$x = A e^{t/4} \cos(t - \phi)$$

These oscillate under an exponentially growing envelope. The derivative does the same, but is off phase. The result is that the trajectory traced out by  $(x,y)$  is an expanding spiral.

[4] It turns out that the same system models the relationship between Romeo and Juliet. The MIT Humanities Department has analyzed the plot of Shakespeare's play and found the following. If  $R$  denotes Romeo's love for Juliet, and  $J$  denotes Juliet's love for Romeo, then

$$R' = J$$

$$J' = -(17/16)R + (1/2)J$$

Romeo is a puppy dog. He has little selfawareness; the change in his feelings towards Juliet has nothing to do with how he himself feels at the moment; it is completely dependent on how she feels about him. Juliet is more complex. She has a healthy self awareness; if she loves him, that very fact causes her to love him more. On the other hand, if he seems to love her, she gets frightened and starts to love him less.

Let's start the action at  $(1,0)$ . So Romeo is fond of Juliet but she is neutral towards him. However, she does notice that he is fond of her, and this makes her somewhat hostile. As she becomes more distant, his affection wanes. Eventually he is neutral and she really doesn't like him. This continues; presently he stays away from her, and this very fact makes her more intested. She warms to him, he notices and his rate of increase of disinterest starts to ameliorate. Eventually she is neutral and just as he bottoms out. He then starts to feel better towards her, but still stays away, and now both his attitude and hers cause her to feel progressively more well disposed towards him. This causes him to continue to warm to her.

Following this around, you wind up at  $J = 0$  again, but now  $R$  has increased. This is a cyclical relationship, but with each cycle the intensity increases. We all know the sad outcome.

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