18.03 Problem Set 4

I encourage collaboration on homework in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

II. Second-order linear equations

L13	W 3 Mar	Forced systems: superposition, Exponential Response Formula, driving through the spring: Notes O.4; EP 2.6, pp. 162–164 (157–159 in 5th ed) only—see SN 7 for beats.
R09	Th 4 Mar	Sinusoidal solutions: amplitude and phase.
L14	F 5 Mar	Polar approach to phase lag, complex gain, driving through the dashpot: SN 14.2, EP 2.6, pp. 167–169 (165–167 in 5th ed).
L15	M 8 Mar	Operators, undetermined coefficients, resonance: SN 10, SN 11, EP 2.5 pp. 148–157 (144–153 in 5th ed).
R10	T 9 Mar	Ditto.
L16	W 10 Mar	Frequency response, RLC circuits: SN 14, SN 8, EP 2.7.
R11	Th 11 Mar	Ditto.
L17	F 12 Mar	LTI systems, superposition, review.
L18	M 15 Mar	Engineering examples: an interview with Prof. Kim Vandiver
R12	T 16 Mar	Exam review.
L19	W 17 Mar	**Hour Exam II.**

Part I.

13. (W 3 Mar) Notes 2-F6bcd.

14. (F 5 Mar) Find the general solution of $\dfrac{d^3 x}{dt^3} - x = e^{2t}$.

15. (M 8 Mar) Notes 2-C8cd; EP 2.5: 2, 8, 11, 14.

16. (W 10 Mar) EP 2.6: 17.

Part II.

13. (W 3 Mar) [Superposition; exponential response formula] Find the general real solution of:

(a) $\dot{x} + 2x = e^{3t}\cos(4t)$. (You could do this using an integrating factor, but it's easier to write the equation as the real part of a complex-valued equation with exponential right hand side, and use the ERF.)

(b) $\ddot{x} + \dot{x} + 2x = \cos t$. (Express the sinusoidal solution of this equation in two ways: as $a\cos t + b\sin t$, and as $A\cos(t - \phi)$.)

Now open the Mathlet `Forced Damped Vibrations`. As in `Damped Vibrations`, the initial conditions are set by the box at left. There is now a forcing term, $A\cos(\omega t)$, and the values of A and ω are adjustable by sliders, as are the values of the mass, damping constant, and spring constant. The force in this system is applied directly to the mass (rather than being mediated by the spring or the dashpot).

Set $m = 1.00$, $b = 1.00$, and $k = 2.00$.

(c) Start by setting $A = 0.00$, so you are looking at the homogeneous case. Set the initial condition to $x(0) = 0$ and $\dot{x}(0) = 1$, select the Solution curve only, and measure the pseudoperiod. (Notice that a slider above the graphing window lets you adjust the vertical scale.) How well does your measurement agree with what you found in **(b)**?

(d) Now set $A = 1.00$ and $\omega = 1.00$. Select the green Steady State curve. Measure its amplitude and the value at $t = 0$. You computed this solution in **(b)**. Compare the computed amplitude and value at $t = 0$ with what you measured. Also compute $\dot{x}_p(0)$.

(e) Finally, select initial condition $x(0) = 0$, $\dot{x}(0) = 0$: so-called *rest initial conditions*. (Clicking on a hashmark gives the exact value.) If you select Steady State, Transient, and Solution, you can see all three: $x = x_p + x_h$. Please compute x_h.

14. (F 5 Mar) [Models and complex gain] This problem employs the Mathlet `Amplitude and Phase: Second Order I`, which illustrates the steady state sinusoidal system response of a spring/mass/dashpot system driven through the spring, as discussed in Lecture 13. The amplitude of the input signal is 1, so the amplitude of the system response is the gain. Select $b = 0.5$, $k = 4.00$, and $\omega = 2.00$. Animate the system.

(a) Verify the displayed values of time lag and gain.

(b) The input signal in this system is the position $y(t) = \cos(\omega t)$ of the top of the spring (the cyan box). The system response is the position of the mass (the yellow box), and we are looking just at the steady state solution $x(t)$. Determine the complex gain for this system. From it determine the gain of the system, as a function of ω. Finally determine $\tan\phi$ where ϕ is the phase lag.

15. (M 8 Mar) [Operators, resonance, undetermined coefficients] Find the general real solution of:

(a) $\dot{x} + x = e^{-t}$. (Use the Resonant ERF.)

(b) $\dfrac{d^3x}{dt^3} - \dfrac{dx}{dt} = t^2 + 1$.

(c) $2\ddot{x} + \dot{x} = (t^2 + 1)e^{2t}$.

16. (W 10 Mar) [Frequency response] This problem will use the applet `Amplitude and Phase: Second Order I` again. Set $k = 4.00$, $b = 0.50$, as in **II.14**. These settings will be in force for parts **(a)** through **(c)**.

(a) In **II.14** you studied the response of this system when $\omega = 2.00$. The gain is pretty large with that setting. Let's investigate the gain for other values of ω. You can see a graph of the gain as a function of the input circular frequency ω by invoking [`Bode Plots`]. The top window shows the gain as a function of ω, and the bottom window graphs $-\phi$ as a function of ω. Move the ω slider and verify that these readings correspond to the graph of the system response at left. You can see a readout of the value of the gain and the phase lag for the selected value of ω by rolling the cursor over the relevant window.

From your experimentation with the applet, do you believe that the gain maximal for $\omega = 2.00$, or is the "practical resonance" peak at a different value of ω?

In **II.14(b)** you wrote down a formula for the gain as a function of ω (for these values of k and b), $g(\omega)$. Now find the value $\omega = \omega_r$ which maximizes $g(\omega)$. (Hint: it'll be easier to minimize the square of the denominator.) Is it at $\omega = 2$? Finally, what is the maximal gain?

(b) Experiment to find the value of ω giving phase lag as close to $45°$ as you can. In **II.14(b)** you also gave a formula for $\tan \phi$. Determine the positive value of ω for which the phase lag equals $45°$. Compare.

(c) Now invoke the [`Nyquist Plot`]. This shows the trajectory of the complex gain $H(\omega)$ as ω runs from 0 to ∞. The value of $H(\omega)$ corresponding to the selected value of ω is shown as a yellow diamond. This means that the length of the yellow strut equals the gain, and the size of the green arc equals the phase lag. Again grab the ω slider and move it slowly from 0 to 4. Please submit a sketch of the Nyquist plot with ω such that $\phi(\omega)$ is as close to $\frac{\pi}{4}$ as you can get it.

(d) Finally, set $\omega = 2$ and leave $k = 4.00$, but adjust the value of b by grabbing the b slider. What do you observe about the position of the yellow strut in the Nyquist plot? Try setting k to a different value, and adjust ω so that the phase lag is close to $\frac{\pi}{2}$. Now vary b and comment on what happens to the phase lag. Please explain this observation as follows. Write down a formula for the complex gain $H(\omega)$ for general values of b, k, and ω. What does $\phi = \frac{\pi}{2}$ say about the complex gain? Finally, what relationship does this imply about b, k, and ω? Does this relationship explain your observation?

Part I answers. 14. $\frac{1}{7}e^{2t} + c_1 e^t + e^{-t/2}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$ **or** $\frac{1}{7}e^{2t} + c_1 e^t + Ae^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t - \phi\right)$.

18.03 Differential Equations
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