# Problems: Applications of Spherical Coordinates

Find the average distance of a point in a solid sphere of radius $a$ from:

(a) the center,

(b) a fixed diameter, and

(c) a fixed plane through the center.

**Answer:** Recall that the average value of a function $f(x, y, z)$ over a volume $D$ is given by $\frac{1}{V}\iiint_D f(x, y, z)\, dV$. We know $V = \frac{4}{3}\pi a^3$. For each of these problems, we'll assume $D$ is a sphere centered at the origin.

(a) In this case, $f(x, y, z) = \rho$ and so:

$$\begin{aligned} \text{A.V.} &= \frac{1}{4\pi a^3/3}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \frac{1}{4\pi a^3/3}\int_0^{2\pi}\int_0^{\pi} \frac{1}{4}a^4 \sin\phi\, d\phi\, d\theta \\ &= \frac{3a}{16\pi}\int_0^{2\pi} 2\, d\theta \\ &= \frac{3a}{8\pi}(2\pi) = 3a/4. \end{aligned}$$

(b) Here we'll use the $z$-axis as the diameter in question, in which case $f = r = \rho\sin\phi$.

$$\begin{aligned} \text{A.V.} &= \frac{1}{4\pi a^3/3}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \sin^2\phi\, d\rho\, d\phi\, d\theta \\ &= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi} \frac{a^4}{4} \sin^2\phi\, d\phi\, d\theta \\ &= \frac{3a}{16\pi}\int_0^{2\pi} \frac{\pi}{2}\, d\theta \\ &= \frac{3a}{32}\cdot(2\pi) = 3\pi a/16. \end{aligned}$$

(c) If we choose the $xy$-plane, $f = |z|$. Because spheres are symmetric the average value of the upper half will equal the average value over the whole sphere, so we compute just that ($V = \frac{2}{3}\pi a^3$).

$$\begin{aligned} \text{A.V.} &= \frac{3}{2\pi a^3}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} \rho^3 \cos\phi\sin\phi\, d\rho\, d\phi\, d\theta \\ &= \frac{3}{2\pi a^3}\int_0^{2\pi}\int_0^{\pi/2} \frac{a^4}{4} \cos\phi\sin\phi\, d\phi\, d\theta \\ &= \frac{3a}{8\pi}\int_0^{2\pi} \frac{1}{2}\, d\theta \\ &= \frac{3a}{16\pi}(2\pi) = 3a/8. \end{aligned}$$