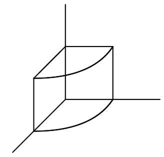
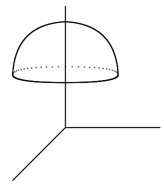
18.02 Exam 4 – Solutions

Problem 1.



$$\int_0^{\pi/2}\int_0^1\int_0^1 r^2\, r\, dz\, dr\, d\theta.$$

Problem 2.



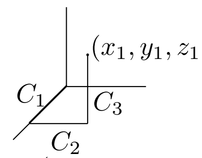
a) sphere: $\rho = 2a\cos\phi$. b) plane: $\rho = a\sec\phi$.

c) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_{a\sec\phi}^{2a\cos\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$

Problem 3.

a) $\frac{\partial}{\partial y}(2xy+z^3) = 2x = \frac{\partial}{\partial x}(x^2+2yz)$; $\frac{\partial}{\partial z}(2xy+z^3) = 3z^2 = \frac{\partial}{\partial x}(y^2+3xz^2-1)$;
$\frac{\partial}{\partial z}(x^2+2yz) = 2y = \frac{\partial}{\partial y}(y^2+3xz^2-1)$; so $\vec F$ is conservative.

b) <u>Method 1:</u> $f(x,y,z) = \int_{C_1+C_2+C_3}\vec F\cdot d\vec r$;



$\int_{C_1}\vec F\cdot d\vec r = \int_0^{x_1}(2xy+z^3)\,dx = \int_0^{x_1} 0\,dx = 0 \quad (y=0,\ z=0)$
$\int_{C_2}\vec F\cdot d\vec r = \int_0^{y_1}(x^2+2yz)\,dy = \int_0^{y_1} x_1^2\,dy = x_1^2 y_1 \quad (x=x_1,\ z=0)$
$\int_{C_3}\vec F\cdot d\vec r = \int_0^{z_1}(y^2+3xz^2-1)\,dz = \int_0^{z_1}(y_1^2+3x_1z^2-1)\,dz = y_1^2z_1 + x_1z_1^3 - z_1 \quad (x=x_1,\ y=y_1)$
So $f(x,y,z) = x^2y + y^2z + xz^3 - z + c$.

<u>Method 2:</u> $\frac{\partial f}{\partial x} = 2xy+z^3$, so $f(x,y,z) = x^2y + xz^3 + g(y,z)$.
$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2+2yz$, so $\frac{\partial g}{\partial y} = 2yz$.
Therefore $g(y,z) = y^2z + h(z)$, and $f(x,y,z) = x^2y + xz^3 + y^2z + h(z)$.
$\frac{\partial f}{\partial z} = 3xz^2 + y^2 + h'(z) = y^2 + 3xz^2 - 1$, so $h'(z) = -1$.
Therefore $h(z) = -z + c$, and $f(x,y,z) = x^2y + xz^3 + y^2z - z + c$.

Problem 4.

a) S is the graph of $z = f(x,y) = 1 - x^2 - y^2$, so $\hat n\, dS = \langle -f_x, -f_y, 1\rangle\, dA = \langle 2x, 2y, 1\rangle\, dA$.
Therefore $\iint_S \vec F\cdot\hat n\, dS = \iint_S \langle x, y, 2(1-z)\rangle\cdot\langle 2x, 2y, 1\rangle\, dA = \iint_S 2x^2+2y^2+2(1-z)\, dA = \iint_S 4x^2 + 4y^2\, dA$ (since $z = 1-x^2-y^2$).
Shadow = unit disc $x^2+y^2 \le 1$; switching to polar coordinates, we have
$\iint_S \vec F\cdot\hat n\, dS = \int_0^{2\pi}\int_0^1 4r^2\, r\, dr\, d\theta = \int_0^{2\pi}\left[r^4\right]_0^1 d\theta = 2\pi$.

b) Let $T =$ unit disc in the xy-plane, with normal vector pointing down ($\hat n = -\hat k$). Then
$\iint_T \vec F\cdot\hat n\, dS = \iint_T \langle x, y, 2\rangle\cdot(-\hat k)\, dS = \iint_T -2\, dS = -2\,\text{Area} = -2\pi$. By divergence theorem,
$\iint_{S+T}\vec F\cdot\hat n\, dS = \iiint_D \operatorname{div}\vec F\, dV = 0$, since $\operatorname{div}\vec F = 1+1-2 = 0$. Therefore $\iint_S = -\iint_T = +2\pi$.

Problem 5.

a) $\operatorname{curl}\vec F = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \partial_x & \partial_y & \partial_z \\ -2xz & 0 & y^2 \end{vmatrix} = 2y\hat\imath - 2x\hat\jmath.$

b) On the unit sphere, $\hat n = x\hat\imath + y\hat\jmath + z\hat k$, so $\operatorname{curl}\vec F\cdot\hat n = \langle 2y, -2x, 0\rangle\cdot\langle x, y, z\rangle = 2xy - 2xy = 0$; therefore $\iint_R \operatorname{curl}\vec F\cdot\hat n\, dS = 0$.

c) By Stokes, $\oint_C \vec F\cdot d\vec r = \iint_R \operatorname{curl}\vec F\cdot\hat n\, dS$, where R is the region delimited by C on the unit sphere. Using the result of b), we get $\oint_C \vec F\cdot d\vec r = \iint_R \operatorname{curl}\vec F\cdot\hat n\, dS = 0$.