Problems: Mass and Average Value

Let R be the quarter of the unit circle in the first quadrant with density $\delta(x, y) = y$.

1. Find the mass of R.

Because R is a circular sector, it makes sense to use polar coordinates. The limits of integration are then $0 \le r \le 1$ and $0 \le \theta \le \pi/2$. In addition we have $\delta = r \sin\theta$. To find the mass of the region, we integrate the product of density and area.

$$\begin{aligned} M &= \iint_R \delta\, dA \\ &= \int_0^{\pi/2} \int_0^1 (r \sin\theta)\, r\, dr\, d\theta \\ &= \int_0^{\pi/2} \int_0^1 r^2 \sin\theta\, dr\, d\theta. \end{aligned}$$

Inner: $\frac{1}{3} r^3 \sin\theta \Big|_0^1 = \frac{1}{3} \sin\theta$.
Outer: $-\frac{1}{3} \cos\theta \Big|_0^{\pi/2} = \frac{1}{3}$.
The region has mass 1/3.

This seems like a reasonable conclusion – the region has area a little greater than 1/2 and average density around 1/2.

2. Find the center of mass.

The center of mass (x_{cm}, y_{cm}) is described by

$$x_{cm} = \frac{1}{M} \iint_R x\delta\, dA \quad \text{and} \quad y_{cm} = \frac{1}{M} \iint_R y\delta\, dA.$$

From (1), $M = \frac{1}{3}$.

$$\begin{aligned} x_{cm} &= \frac{1}{M} \iint_R x\delta\, dA \\ &= 3 \int_0^{\pi/2} \int_0^1 (r \cos\theta)(r \sin\theta) r\, dr\, d\theta \\ &= \int_0^{\pi/2} \int_0^1 3r^3 \cos\theta \sin\theta\, dr\, d\theta. \end{aligned}$$

Inner: $\frac{3}{4} r^4 \cos\theta \sin\theta \Big|_0^1 = \frac{3}{4} \cos\theta \sin\theta$.

Outer: $\frac{3}{4} \frac{1}{2} (\sin\theta)^2 \Big|_0^{\pi/2} = \frac{3}{8} = x_{cm}$.

$$
\begin{aligned}
y_{cm} &= \frac{1}{M}\iint_R y\delta \, dA \\
&= 3\int_0^{\pi/2}\int_0^1 (r\sin\theta)(r\sin\theta)r\,dr\,d\theta \\
&= \int_0^{\pi/2}\int_0^1 3r^3\sin^2\theta\,dr\,d\theta.
\end{aligned}
$$

Inner: $\left.\frac{3}{4}r^4\sin^2\theta\right|_0^1 = \frac{3}{4}\sin^2\theta.$

Outer: $\left.\frac{3}{4}\left(\frac{\theta}{2}-\frac{1}{4}\sin(2\theta)\right)\right|_0^{\pi/2} = \frac{3\pi}{16} = y_{cm}.$

The center of mass is at $\left(\frac{3}{8},\frac{3}{4}\right) \approx (.4, .6)$.

This point is within R and agrees with our intuition that $x_{cm} < 1/2$ and $y_{cm} > x_{cm}$.

3. Find the average distance from a point in R to the x axis.

To find the average of a function $f(x,y)$ over an area, we compute $\frac{1}{\text{Area}}\iint_R f(x,y)\,dA$. Here $f(x,y) = y$.

$$
\begin{aligned}
\frac{1}{\text{Area}}\iint_R y\,dA &= \frac{1}{\pi/4}\int_0^{\pi/2}\int_0^1 (r\sin\theta)r\,dr\,d\theta \\
&= \frac{4}{\pi}\int_0^{\pi/2}\int_0^1 r^2\sin\theta\,dr\,d\theta.
\end{aligned}
$$

This should look familiar – we computed in (1) that $\int_0^{\pi/2}\int_0^1 r^2\sin\theta\,dr\,d\theta = \frac{1}{3}$. The average distance from a point in R to the x axis is $\frac{4}{\pi}\cdot\frac{1}{3} = \frac{4}{3\pi}$.

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18.02SC Multivariable Calculus
Fall 2010

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