Problems: Chain Rule Practice

One application of the chain rule is to problems in which you are given a function of x and y with inputs in polar coordinates. For example, let $w = (x^2 + y^2)xy$, $x = r\cos\theta$ and $y = r\sin\theta$.

1. Use the chain rule to find $\dfrac{\partial w}{\partial r}$.

<u>Answer:</u> We apply the chain rule.

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= (3x^2y + y^3)\cos\theta + (3xy^2 + x^3)\sin\theta.\end{aligned}$$

We could check our work by substituting to get $w = r^4\cos\theta\sin\theta$ and calculating $\frac{dw}{dr}$ directly. We practice using the chain rule because such substitutions are not always practical.

2. Find the total differential dw in terms of dr and $d\theta$.

<u>Answer:</u> We know $dw = w_x\,dx + w_y\,dy$. In terms of r and θ, $dx = x_r\,dr + x_\theta\,d\theta = \cos\theta\,dr - r\sin\theta\,d\theta$. Similarly, $dy = \sin\theta\,dr + r\cos\theta\,d\theta$. Thus,

$$\begin{aligned}dw &= w_x(\cos\theta\,dr - r\sin\theta\,d\theta) + w_y(\sin\theta\,dr + r\cos\theta\,d\theta) \\ &= (w_x\cos\theta + w_y\sin\theta)\,dr + (w_y r\cos\theta - w_x r\sin\theta)\,d\theta.\end{aligned}$$

We could stop here or go on to compute:

$$\begin{aligned}dw &= [(3x^2y + y^3)\cos\theta + (3xy^2 + x^3)\sin\theta]\,dr + [(3xy^2 + x^3)r\cos\theta - (3x^2y + y^3)r\sin\theta]\,d\theta \\ &= 4r^3\cos\theta\sin\theta\,dr + r^4(\cos^2\theta - \sin^2\theta)\,d\theta.\end{aligned}$$

Note that the answer to (1) appears in the dr component of dw. In practice, the best format for the answer is the one that is easiest to use.

3. Find $\dfrac{\partial w}{\partial r}$ at the point $(r, \theta) = (2, \pi/4)$.

<u>Answer:</u> Recall that $\frac{\partial w}{\partial r} = (3x^2y + y^3)\cos\theta + (3xy^2 + x^3)\sin\theta$. We need only compute $x = \sqrt{2}$ and $y = \sqrt{2}$ and plug in values.

$$\begin{aligned}(3x^2y + y^3)\cos\theta + (3xy^2 + x^3)\sin\theta &= 4(\sqrt{2})^3(\frac{\sqrt{2}}{2}) + 4(\sqrt{2})^3(\frac{\sqrt{2}}{2}) \\ &= 16.\end{aligned}$$