18.02 Problem Set 4, Part II Solutions

1. (a) The graphs of $x \to F_2(x,t) = \cos^2(x-2t)$ for $t = -1, 0, 1$ all have the same sinusoidal shape $f(u) = \cos^2(u)$ shifted along the x-axis.
(b) This would represent the string displaced into the shape f and then this wave form traveling down the string over time with the 'wave speed' $= 2$ (linear units/unit time). In physics this is called a traveling wave (not surprisingly).
The applet shows the same shape f translated along the y (= time) axis – that is, if you take a trace curve on the surface in any plane y = constant, you get one of the wave forms f shifted along the x-direction. (Note that the surface graph in 3D appears static, until one remembers that the y-axis represents time here; in the language of physics, this would be called a graph in the space-time domain.)

2. We have two surfaces defined by

$$z = f(x,y) = x^2 - y^2$$

$$z = g(x,y) = 2 + (x-y)^2.$$

a. Let (x,y,z) be in both surfaces. Then $z = f(x,y)$ and $z = g(x,y)$ which gives

$$x^2 - y^2 = 2 + (x-y)^2$$

or

$$x^2 - y^2 = 2 + x^2 - 2xy + y^2$$

which reduces to

$$-2y^2 + 2xy = 2$$

or

$$x = y + \frac{1}{y}$$

assuming $y \neq 0$.

When one does an intersection problem, it is possible to get extraneous solutions. Let's plug back in our formula for x and see if all the points we found do give rise to common points between surfaces f and g.

$$x^2 - y^2 = (y + y^{-1})^2 - y^2 = 2 + y^{-2} = 2 + ((y + y^{-1}) - y)^2 = 2 + (x-y)^2$$

So this checks out. To parameterize our curve, we choose $y = t$ and then get

$$\begin{aligned} x &= t + t^{-1} \\ y &= t \\ z &= 2 + t^{-2} \end{aligned}$$

(b) First we find a normal to the plane T_1 tangent to surface f at $(2, 1, 3)$:

$$\begin{aligned} f_x &= 2x \\ f_y &= -2y \\ f_x(2,1) &= 4 \\ f_y(2,1) &= -2. \end{aligned}$$

We may then use the formula for the normal

$$\vec{n}_1 = \langle f_x(2,1), f_y(2,1), -1 \rangle = \langle 4, -2, -1 \rangle .$$

We find a normal to the plane T_2 tangent to the surface g at $(2, 1, 3)$ by the same method:

$$\begin{aligned} g_x &= 2(x-y) \\ g_y &= -2(x-y) \\ g_x(2,1) &= 2 \\ g_y(2,1) &= -2. \end{aligned}$$

The normal is

$$\vec{n}_2 = \langle g_x(2,1), g_y(2,1), -1 \rangle = \langle 2, -2, -1 \rangle .$$

Then

$$\measuredangle(T_1, T_2) = \measuredangle(\vec{n}_1, \vec{n}_2) = \theta$$

where

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{8+4+1}{\sqrt{21}\sqrt{9}} = \frac{13}{3\sqrt{21}}.$$

So

$$\theta = \cos^{-1}\left(\frac{13}{3\sqrt{21}}\right) \approx .33\text{rad} \approx 19\text{deg}.$$

(c) $\vec{r}(t) = \langle t + t^{-1}, t, 2 + t^{-2} \rangle$. So

$$\vec{r}'(t) = \left\langle 1 - t^{-2}, 1, -2t^{-3} \right\rangle .$$

The point $P_0 = (2,1,3) = \vec{r}(t)$ for $t = 1$. So the velocity vector of the parameterization as it passes through P_0 is

$$\vec{r}'(1) = \langle 0, 1, -2 \rangle .$$

We think of this vector as being based at point P_0, pointing along the curve $\vec{r}$. Given this, we know its initial point lies in the planes T_1, T_2. What remains is to prove that the vector is parallel to both planes. We check this using our normal vectors:

$$\begin{aligned}\vec{n}_1 \cdot \vec{r}'(1) &= \langle 4, -2, -1 \rangle \cdot \langle 0, 1, -2 \rangle = 0. \\ \vec{n}_2 \cdot \vec{r}'(1) &= \langle 2, -2, -1 \rangle \cdot \langle 0, 1, -2 \rangle = 0.\end{aligned}$$

3. The contour plot is a set of circles centered at the origin, with the f-level decreasing as the radius increases. The parabola C_2 is tangent to the level curve $f = \frac{16}{13}$ at the point $(0, \frac{3}{2})$, and to the level curve $f = \frac{16}{9}$ at the points $(\pm 1, \frac{1}{2})$.
b) $x = t, \quad y = 1.5 - t^2, \quad z = f(x(t), y(t)) = \frac{4}{1+t^2+(1.5-t^2)^2}$.
d) Computing $\frac{dz}{dt} = \frac{d}{dt}\left(\frac{4}{1+t^2+(1.5-t^2)^2}\right)$ and setting the result equal to zero gives $4t(t^2-1) = 0$. Critical points are thus at $t = 0$ and $t = \pm 1$, which gives the points $(0, \frac{3}{2}, \frac{16}{13})$, which is a local min, and $(\pm 1, \frac{1}{2}, \frac{16}{9})$, which are local max's on the surface S.
e) We're looking for the max/min's of distance^2 $= t^2 + (\frac{3}{2} - t^2)^2$. Differentiating and setting equal to zero gives the same equation as in part(d): $4t(t^2-1) = 0$.
Geometrically, the reason that you get the same results is that the surface given by $z = f(x,y)$ decreases symmetrically as (x,y) moves away from the origin. The point $(0, \frac{3}{2})$ gives a local min on f, since its distance from O is a local max; and the points $(\pm 1, \frac{1}{2})$ give local max's on f, since their distance from O is a local min.
This is confirmed by surface and curve graphs, and also by the level curve picture.

4. We are considering the sum S, writable as the function

$$f(x, y, z) = x^3 + y^3 + z^3$$

on the set of (x, y, z) satisfying $x^2 + y^2 + z^2 = 27$; $x, y, z \geq 0$. Geometrically this is the part of a sphere lying in the first octant. Algebraically, we see that we only need to work with two variables; the variable z can be solved for in terms of the other two.

$$z = \sqrt{27 - x^2 - y^2}.$$

Here we limit x, y to a quadrant Q of a disc: $x, y \geq 0, x^2 + y^2 \leq 27$. We may therefore write our function f in terms of just x, y:

$$f(x, y) = x^3 + y^3 + (27 - x^2 - y^2)^{3/2}.$$

Partial derivatives are

$$f_x(x, y) = 3x^2 + \frac{3}{2}(-2x)\sqrt{27 - x^2 - y^2}$$

and

$$f_y(x, y) = 3y^2 + \frac{3}{2}(-2y)\sqrt{27 - x^2 - y^2}.$$

Critical points occur when $\langle f_x(x, y), f_y(x, y)\rangle = \langle 0, 0\rangle$. Looking at the equations we see

$$x = 0, \qquad \text{or } x = \sqrt{27 - x^2 - y^2}$$

and

$$y = 0, \qquad \text{or } y = \sqrt{27 - x^2 - y^2}.$$

We have two independent choices; this gives four possibilities, which work out to $(0, 0), (0, \sqrt{27/2}), (\sqrt{27/2}, 0), (3, 3)$.

2nd derivative test:

$$\begin{aligned}
f_{xx} &= 6x - 3(27 - x^2 - y^2)^{\frac{1}{2}} + 3x^2(27 - x^2 - y^2)^{-\frac{1}{2}} \\
f_{xy} &= f_{yx} = 3xy(27 - x^2 - y^2)^{-\frac{1}{2}} \\
f_{yy} &= 6y - 3(27 - x^2 - y^2)^{\frac{1}{2}} + 3y^2(27 - x^2 - y^2)^{-\frac{1}{2}}
\end{aligned}$$

At (0,0);

$$A = f_{xx}(0, 0) = -9\sqrt{3},\ B = f_{xy}(0, 0) = 0,\ C = f_{yy}(0, 0) = -9\sqrt{3}.$$

Therefore, $AC - B^2 = 243 > 0$ and $A < 0$, which implies the critical point is a relative maximum. $S = 81\sqrt{3}$.

At $(0, 3\sqrt{\frac{3}{2}})$ and $(3\sqrt{\frac{3}{2}}, 0)$.

We compute $A = 18\sqrt{3/2}$, $B = 0$, $C = -9\sqrt{3/2}$. Therefore, $AC - B^2 < -$, which means we have a saddle points at $(3\sqrt{\frac{3}{2}}, 0, 3\sqrt{\frac{3}{2}})$ and $(0, 3\sqrt{\frac{3}{2}}, 3\sqrt{\frac{3}{2}})$, neither max nor min.

At (3,3) we compute $A = 18 = C$ and $B = 9 \Rightarrow AC - B^2 = 243$, since $A > 0$ this is a minimum $\Rightarrow$ (3,3,3) is a relative minimum. $S = 3 \cdot 3^3 = 81$.

Boundary test: $x^2 + y^2 = 27$ is the boundary of the region where f is defined. Parametrize by $x = 3\sqrt{3}\cos t$, $y = 3\sqrt{3}\sin t$, so $f(3\sqrt{3}\cos t, 3\sqrt{3}\sin t) = 27(\cos^3 t + \sin^3 t)$ (since $z = 0 = (27 - x^2 - y^2)^{1/2}$). max/min by 1-variable calculus: $\frac{d}{dt} 27(\cos^3 t + \sin^3 t) = 81\ \cos t\ \sin t\,(\sin t - \cos t)$.
Critical points: $t = 0, \frac{\pi}{4}, \frac{\pi}{2} \ldots$.
Observe that the derivative changes its sign from $+ \to -$ at $t = 0$, from $- \to +$ at $t = \frac{\pi}{4}$, and from $+ \to -$ at $t = \frac{\pi}{2}$.

We get relative maxima at $t = 0$, $x = 3\sqrt{3}$, $y = 0$, $z = 0$, and $t = \frac{\pi}{2}$, $x = 0$, $y = 3\sqrt{3}$, $z = 0$. For $t = \frac{\pi}{4}$ we have a relative minimum with $x = 3\sqrt{6}/2$, $y = 3\sqrt{6}/2$, $z = 0$. Note that other critical values of t give the same or negative values, so these suffice. The value at the relative maxima on the boundary is $S = 81\sqrt{3}$, and for the relative minimum it is $S = 81\sqrt{\frac{3}{2}}$.

Conclusion: Largest $S = 81\sqrt{3}$: just one number greater than 0, equal to $3\sqrt{3}$.
Smallest $S = 81$: three equal numbers, equal to 3.

5.(a) We want the critical points of $f(\alpha, \beta) = \cos\alpha\ \cos\beta\ \cos(\alpha + \beta)$, where α and β are in the range $[0, \frac{\pi}{2}]$. We take the first partials of f and set them equal to zero.

$f_\alpha(\alpha, \beta) = -\sin\alpha\ \cos\beta\ \cos(\alpha + \beta) - \cos\alpha\ \cos\beta\ \sin(\alpha + \beta) = 0$ and
$f_\beta(\alpha, \beta) = -\cos\alpha\ \sin\beta\ \cos(\alpha + \beta) - \cos\alpha\ \cos\beta\ \sin(\alpha + \beta) = 0.$

Using the sine addition formula $\sin(a + b) = \sin a \cos b + \cos a \sin b$ we get

$f_\alpha(\alpha, \beta) = -\cos\beta\ \sin(2\alpha + \beta) = 0$ and
$f_\beta(\alpha, \beta) = -\cos\alpha\ \sin(\alpha + 2\beta) = 0.$

One solution is $\alpha = \beta = \pi/2$, but this gives $f = 0$, which is not the largest negative component. $\alpha = \pi/2$ and $\beta \neq \pi/2$ gives a contradiction, as does $\alpha \neq \pi/2$ and $\beta = \pi/2$ (show this). Then $\alpha \neq \pi/2$ and $\beta \neq \pi/2$ gives $\sin(\alpha + 2\beta) = 0$ and $\sin(2\alpha + \beta) = 0$, which implies that

$\alpha + 2\beta = \pi$ and $2\alpha + \beta = \pi$. Solving, we get $\alpha = \beta = \dfrac{\pi}{3}$.
Second-derivative test to show that this is in fact a minimum (i.e., most negative) – optional.

(b) $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -\frac{1}{8}$. Since the length of the wind vector $\mathbf{w} = \langle 1, 0 \rangle$ is 1, this means that one can capture at most $\frac{1}{8}$ or 12.5 % of the force of the wind for the purpose of tacking into the wind.

Suggested Experiments. When you move from $(0,0)$ you will observe

direction	f_x	f_y
E	decreases	stays zero
NE	decreases	increases
N	stays zero	increases
NW	increases	increases
W	increases	stays zero
SW	increases	decreases
S	stays zero	decreases
SE	decreases	decreases

Hiking W or E you descend more and more steeply.

Hiking N or S you ascend more and more steeply.

18.02SC Multivariable Calculus
Fall 2010