JOEL LEWIS: Hi. Welcome back to recitation. You've been learning in lecture about matrices and their various applications, and one of them is to solving systems of linear equations. So I have here a system of three linear equations for you. 2x plus c*z equals 4, x minus y plus 2z equals pi, and x minus 2y plus 2z equals minus 12. So what I'd like you to do is the following.

Find the value of c-- or all values of c-- for which, first of all, there's a unique solution to this system. Second of all, for which the corresponding homogeneous system has a unique solution. So remember that the corresponding homogeneous system is the system where you just replace these constants on the right by 0. So it's a very similar-looking system. The left-hand sides are all the same, but the right-hand sides are replaced with 0. So you want to find the value of c for which this system has a unique solution, the value of c for which the corresponding homogeneous system has a unique solution, and also the values of c for which the corresponding homogeneous system has infinitely many solutions.

Note that I'm not asking you to solve this system of equations, although you're welcome to do so if you like. Although, of course, whether you can or not might depend on the value of c. So why don't you pause the video, take a little while to work out the solutions to these three questions, come back, and we can work it out together.

So hopefully you have some luck working out these problems. Let's start working through them together. So I'm actually going to take parts a and b together at the same time.

And the reason that I'm going to do that is that one thing you've learned is that a system has a unique solution for, on the right-hand side-- sorry-- a system has a unique solution, like this, a square system of linear equations has a unique solution if and only if it has a unique solution regardless of what the right-hand side is. So in particular, the answer to a and the answer to b are exactly the same.

So values of c for which this system has a unique solution are exactly the same as values of c for which the homogeneous system has a unique solution. Now the solutions will be different, of course. But the value of c-- or the values of c-- that make it solvable uniquely, make it solvable uniquely for all right-hand sides.

And so which values of c are those? Well, those are the values of c for which the coefficient matrix on the left-hand side is invertible. So if the coefficient matrix on the left-hand side is invertible, then we can solve this system and we get a unique solution. If it's not invertible, then either we can't solve this system-- like, there are no solutions-- or we can solve this system, but there are infinitely many solutions.

So in both questions a and b, we're asking for the value of c for which the coefficient matrix of the left-hand side is invertible, and that will be when we have a unique solution. So how do we know when a matrix is invertible? Well, let's write down what the matrix is first of all.

So this matrix M that we're after is equal to the matrix 2, 0, c; 1, minus 1, 2; 1, minus 2, 2. So this is the coefficient matrix M of that system, and we want to know for which values of c is it invertible.

Well, when is a matrix invertible? A matrix is invertible-- square matrix is invertible-- precisely when it has non-zero determinant. So we just need to look at the determinant of this matrix. So you've learned how to compute determinants of matrices, I think.

So let's, in this case, we have the det M. So it's a sum or difference of six different terms, and you could get it, for example, by the Laplace expansion if you wanted to. So I'm just going to write out what the six terms are. So it's 2 times minus 1 times 2, plus 0 times 2 times 1, plus c times 1 times minus 2, minus c times minus 1 times 1, minus 2 times minus 2 times 2, minus 0 times 1 times 2. So this is the determinant of this matrix.

You can get it either just by remembering which terms are which and which get a plus sign and which get a minus sign, or by doing the Laplace expansion, or by whatever other tricks you might happen to know. So now we need to know whether or not this determinant is 0. So let's work out what this is.

So this is-- let me start simplifying it. So this is minus 4 plus 0 minus 2c-- this is minus minus c, so plus c-- this is minus minus 8, so plus 8, which is equal to 4 minus c. So the determinant-- right, two of those terms are 0, and so I just get to leave them out. So the determinant of this matrix is 4 minus c. And what we're interested in is when this determinant is non-zero.

So in particular, for c not equal to 0-- sorry, for c not equal to 4-- when c is not 4, the determinant of M is not 0. So when c is not 4, determinant of M is not 0, so both systems-- both the original system and the corresponding homogeneous system-- have a unique solution. So when c is not 4-- so for most values of c-- the determinant is not 0, and the system has a unique solution.

So when c is equal to 4, what happens? Well, when c is equal to 4, we're in the bottom case. We're in the case where the homogeneous system has infinitely many solutions. OK? So let me write that over here.

When c equals 4-- I'm going to abbreviate again-- the homogeneous system has-- I'm going to use this symbol-- this sort of sideways eight symbol means infinity, so I'm going to use it for infinitely many solutions. So when c is 4, the homogeneous system has infinitely many solutions. And you might be curious-- well, so let me say one more thing about that. We know when the coefficient matrix isn't invertible that the system either has zero or infinitely many solutions. But the homogeneous system always has a solution. It always has the solution where everything is all 0. Right? So that's why we know that it's infinitely many here.

And one thing you might ask is can you find any others? Can you find any solutions that aren't just [0, 0, 0]? And the answer is yes. So this is now going beyond when I asked you to do, but I think it's, you know, an interesting thing to see. So if you wanted to find another solution, what do you know? Well, let's go back to the equations that we had.

So when we're dealing with a homogeneous system, the right-hand sides are 0. So I'm just going to cross out these right-hand sides and replace them with 0 so we don't get confused. So this is 0, 0, and 0. So we're dealing with this system: 2x plus c*z equals 0, x minus y plus 2z equals 0, and x minus 2y plus 2z equals 0.

OK, so if you want a solution [x, y, z] to this system, what do you know? Well, from the second equation, you know that the vector [x, y, z] is orthogonal to the vector 1, minus 1, 2. How do you know that? Because this left-hand side, x minus y plus 2z, is equal to [x, y, z] dot 1, minus 1, 2.

And similarly from the third equation, you know that the vector [x, y, z] is orthogonal to the vector 1, minus 2, 2, because this left-hand side is equal to [x, y, z] dot 1, minus 2, 2. Yeah? And that's equal to 0. So from the second and third equations, you know that you're looking for a vector that's orthogonal to both x-- or sorry-- both 1, minus 1, 2, and 1, minus 2, 2.

How do you get a vector perpendicular to two known vectors? Well, you just take their cross product. So let's go back over here. So to find one, you take a cross product of two rows of the coefficient matrix. So in this case, for example, we can take these rows, 1, minus 1, 2; and 1, minus 2, 2. So, for example, the vector 1, minus 1, 2-- OK-- cross the vector 1, minus 2, 2.

Now I've kind of run out of board space, so I'm not going to work out precisely what this vector is for you. But if you like, you can certainly check. You can compute this cross product out with our nice formula for the cross product. It will give you some vector, and then you can check that that vector is indeed a solution of the homogeneous system. So that will give us a second solution of the homogeneous system. Nontrivial we say, because it's not just the 0 solution.

So to quickly recap, we had a system of linear equations. I've now crossed out what the original right-hand side was. We had a system of linear equations, and we were looking for a choice of c for which that system had a unique solution and for which the corresponding homogeneous system had a unique solution. And the values of c that make that work are precisely the values of c such that the coefficient matrix has a non-zero determinant. So that's true for both parts a and b.

And for part c, when we were looking for what values of c give the homogeneous system infinitely many solutions, the answer is any other value of c. Any value of c for which the coefficient matrix does have 0 determinant will give you infinitely many solutions in the homogeneous case, and in non-homogeneous cases will either give you 0 solutions or infinitely many solutions.

And then we also at the end, we briefly discussed one way to find nontrivial solutions in the homogeneous case when there are infinitely many solutions. So I'll end there.