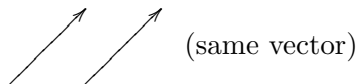# Vectors

Our very first topic is unusual in that we will start with a brief written presentation. More typically we will begin each topic with a videotaped lecture by Professor Auroux and follow that with a brief written presentation.

As we pointed out in the introduction, vectors will be used throughout the course. The basic concepts are straightforward, but you will have to master some new terminology. Another important point we made earlier is that we can view vectors in two different ways: geometrically and algebraically. We will start with the geometric view and introduce terminology along the way.

Geometric view

A vector is defined as having a magnitude and a direction. We represent it by an arrow in the plane or in space. The length of the arrow is the vector's magnitude and the direction of the arrow is the vector's direction.

In this way, two arrows with the same magnitude and direction represent the same vector.

(same vector)

We will refer to the start of the arrow as the *tail* and the end as the *tip* or *head*.

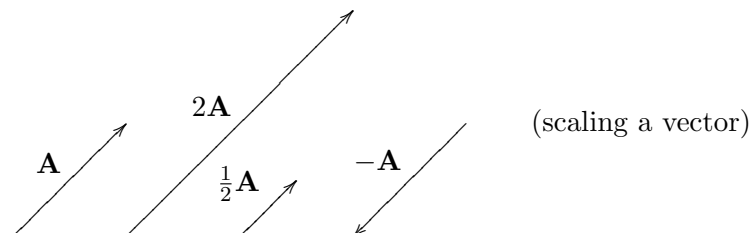The vector between two points will be denoted $\overrightarrow{\mathbf{PQ}}$.
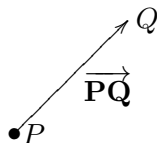
We call P the initial point and Q the terminal point of $\overrightarrow{\mathbf{PQ}}$.

The *magnitude* of the vector $\mathbf{A}$ will be denoted $|\mathbf{A}|$. Magnitude will also be called *length* or *norm*
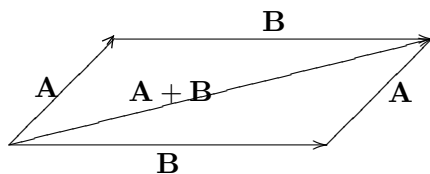
Scaling, adding and subtracting vectors

Scaling a vector means changing its length by a scale factor. For example,

$2\mathbf{A}$ $\mathbf{A}$ $\frac{1}{2}\mathbf{A}$ $-\mathbf{A}$ (scaling a vector)

Because we use numbers to scale a vector we will often refer to real numbers as *scalars*
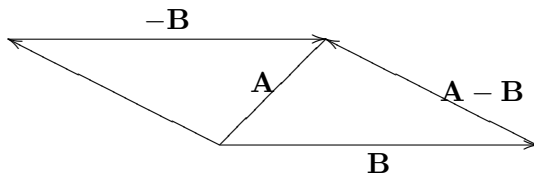
You add vectors by placing them head to tail. As the figure shows, this can be done in

either order

It is often useful to think of vectors as *displacements*. In this way, $\mathbf{A} + \mathbf{B}$ can be thought of as the displacement $\mathbf{A}$ followed by the displacement $\mathbf{B}$.

You subtract vectors either by placing the tail to tail or by adding $\mathbf{A} + (-\mathbf{B})$.

Thought of as displacements $\mathbf{A} - \mathbf{B}$ is the displacement from the end of $\mathbf{B}$ to the end of $\mathbf{A}$.

Algebraic view

As is conventional, we label the origin O. In the plane $O = (0,0)$ and in space $O = (0,0,0)$.

In the xy-plane if we place the tail of $\mathbf{A}$ at the origin, its head will be at the point with coordinates, say, (a_1, a_2). In this way, the coordinates of the head determine the vector $\mathbf{A}$. When we draw $\mathbf{A}$ from the origin we will refer to it as an *origin vector*.
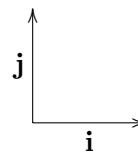
Using the coordinates we write

$$\mathbf{A} = \langle a_1,\, a_2 \rangle.$$

Addition, subtraction and scaling using coordinates is discussed below.

Graphically:
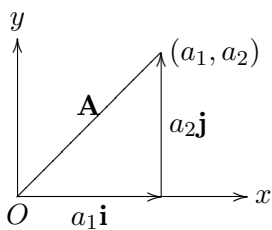
The vectors $\mathbf{i}$ and $\mathbf{j}$ used in the figure above have coordinates $\mathbf{i} = \langle 1,\, 0 \rangle$, $\mathbf{j} = \langle 0,\, 1 \rangle$. We use them so often that they get their own symbols.

Notation and terminology
1. (a_1, a_2) indicates a point in the plane.
2. $\langle a_1,\, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$. This is equal to the vector drawn from the origin to the point (a_1, a_2).
3. For $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$, a_1 and a_2 are called the $\mathbf{i}$ and $\mathbf{j}$ *components* of $\mathbf{A}$. (Note that they are scalars.)
5. $\vec{\mathbf{P}} = \overrightarrow{\mathbf{OP}}$ is the vector from the origin to P.

6. On the blackboard vectors will usually have an arrow above the letter. In print we will often drop the arrow and just use the bold face to indicate a vector, i.e. $\mathbf{P} \equiv \overrightarrow{\mathbf{P}}$.

7. A real number is a *scalar*, you can use it to scale a vector.

Vector algebra using coordinates

For the vectors $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$ we have the following algebraic rules. The figures below connect these rules to the geometric viewpoint.
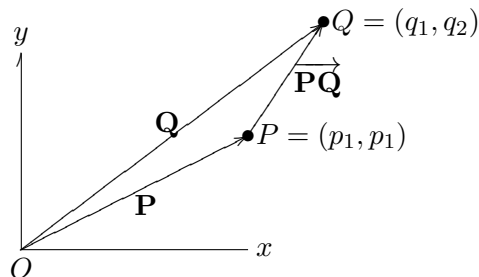
Magnitude: $|\mathbf{A}| = \sqrt{a_1^2 + a_2^2}$ (this is just the Pythagorean theorem)

Addition: $\mathbf{A} + \mathbf{B} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$, that is, $\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$

Subtraction: $\mathbf{A} - \mathbf{B} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j}$, that is, $\langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle$
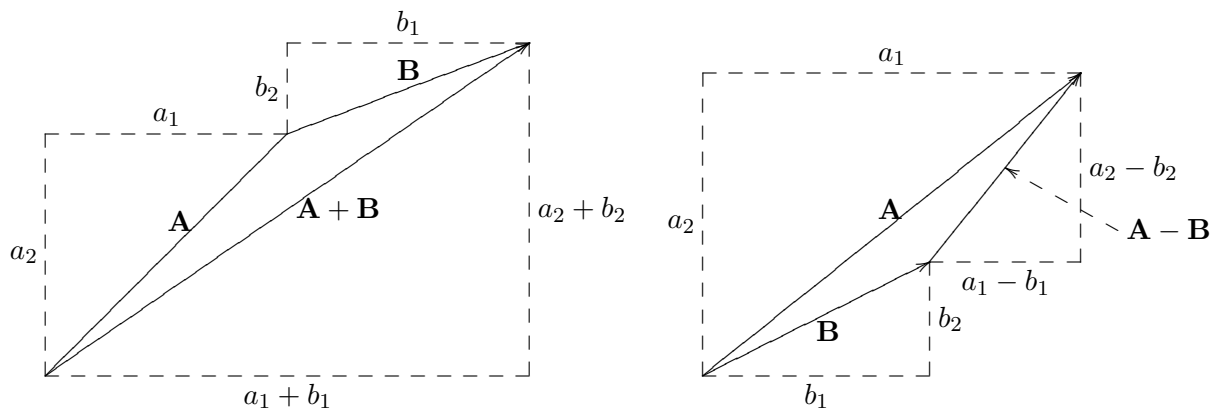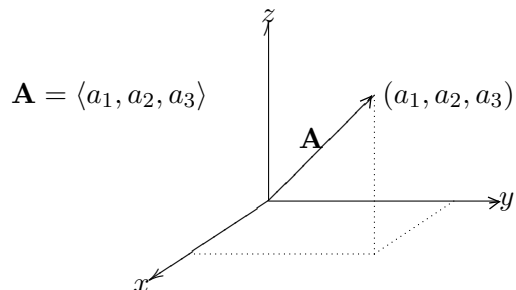

For two points P and Q the vector $\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}}$ i.e., $\overrightarrow{\mathbf{PQ}}$ is the *displacement* from P to Q.

Vectors in three dimensions

We represent a three dimensional vector as an arrow in space. Using coordinates we need three numbers to represent a vector.

Geometrically nothing changes for vectors in three dimensions. They are scaled and added exactly as above.

Algebraically the origin vector $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ starts at the origin and extends to the point (a_1, a_2, a_3). We have the special vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$. Using them

$$\langle a_1, a_2, a_3 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}.$$

Then, for $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$ we have
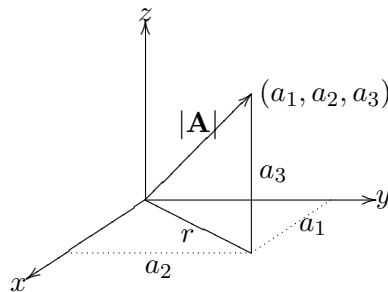
$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle.$$

exactly as in the two dimensional case.

Magnitude in three dimensions also follows from the Pythagorean theorem.

$$|a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}| = |\langle a_1, a_2, a_3 \rangle| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

You can see this in the figure below, where $r = \sqrt{a_1^2 + a_2^2}$ and $|\mathbf{A}| = \sqrt{r^2 + a_3^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Unit vectors

A unit vector is any vector with unit length. When we want to indicate that a vector is a unit vector we put a hat (circumflex) above it, e.g., $\widehat{\mathbf{u}}$.

The special vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors.

Since vectors can be scaled, any vector can be rescaled to be a unit vector.

Example: Find a unit vector that is parallel to $\langle 3, 4 \rangle$.

Answer: Since $|\langle 3, 4 \rangle| = 5$ the vector $\dfrac{1}{5} \langle 3, 4 \rangle = \left\langle \dfrac{3}{5}, \dfrac{4}{5} \right\rangle$ has unit length and is parallel to $\langle 3, 4 \rangle$.

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