EXAM 1 PRACTICE MATERIALS

(1) Let A be an $m \times n$ matrix and r be the rank of A.
  (a) Describe the dimension of the solution space of the equation $A\mathbf{x} = \mathbf{0}$ in terms of m, n, r.
  (b) Suppose there exists $\mathbf{c}$ such that $A\mathbf{x} = \mathbf{c}$ does not have a solution. What can you say about m, n, r?
  (c) If A is invertible, what is the relationship between m, n and r?

(2) Let $\{x_1, x_2, \cdots, x_n\}$ be a basis for the vector space V. Consider the set $\{\sum_{i=1}^n c_{1i}x_i, \cdots, \sum_{i=1}^n c_{ni}x_i\}$ for $c_{ji} \in \mathbb{R}$. Is this still a basis for V? Prove it either way.

(3) Let A, B and C be three vectors (or points) in $\mathbb{R}^3$. Let M be the 3×3 matrix that has A , B and C as its rows (from top to bottom).
  (a) Show that $|\det M| \leq ||A||||B||||C||$.
  (b) Show that if $\{A, B, C\}$ is an orthogonal set then $\det M = \pm ||A||||B||||C||$. When does one get a $+$ and when a $-$?
  (c) Is it true that if $|\det M| = ||A||||B||||C||$ then $\{A, B, C\}$ is orthogonal?

(4) Let L be a map from $\mathbb{R}^3$ to $\mathbb{R}^2$ for which

$$L(u + v) = L(u) + L(v) \qquad (u, v \in \mathbb{R}^3).$$

  (a) Show that $L(nv) = nL(v)$ for any integer n and $v \in \mathbb{R}^3$;
  (b) Show that $L(\frac{1}{n}v) = \frac{1}{n}L(v)$ for any integer n and $v \in \mathbb{R}^3$;
  (c) Show that $L(\frac{m}{n}v) = \frac{n}{m}L(v)$ for any rational number $\frac{n}{m}$ and $v \in \mathbb{R}^3$;
  (d) Conclude that if L is continous, then L must be linear. (We say L is continuous at y if $||L(x) - L(y)|| \to 0$ when $||x - y|| \to 0$.)

(5) Consider the function

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2+y^2} & \text{if } x^2 + y^2 \neq 0, \\ 0 & \text{if } x = y = 0, \end{cases}$$

  (a) Show that the partial derivatives of f are discontinuous at $(0, 0)$;
  (b) Show that the partial derivatives of f are not bounded in any balls around $(0, 0)$;
  (c) Show that f is differentiable at $(0, 0)$.