Solutions for PSet 9

1. (11.9:8) Using Fubini's Theorem (we assumed that the double integral exists):

$$
\begin{aligned}
\int\int_{[0,t]\times[1,t]} \frac{e^{\frac{tx}{y}}}{y^3}\,dx\,dy &= \int_1^t \left(\int_0^t \frac{e^{\frac{tx}{y}}}{y^3}\,dx\right)dy = \\
\int_1^t \left[y^{-3}\frac{y}{t}e^{\frac{tx}{y}}\right]_{x=0}^{t} dy &= \int_1^t \frac{e^{\frac{t^2}{y}}-1}{ty^2}\,dy = \\
\left[-\frac{1}{t^3}e^{\frac{t^2}{y}}+\frac{1}{ty}\right]_{y=1}^{t} &= \frac{1}{t^2}-\frac{1}{t}-\frac{1}{t^3}e^t+\frac{1}{t^3}e^{t^2}
\end{aligned}
$$

2. (11.15:2)

$$
\begin{aligned}
\int\int_S (1+x)\sin y\,dx\,dy &= \int_0^1\left(\int_0^{1+x}(1+x)\sin y\,dy\right)dx = \\
\int_0^1 (1+x)(1-\cos(1+x))\,dx &= \left[\frac{x}{2}(x+2)-(x+1)\sin(x+1)-\cos(x+1)\right]_0^1 \\
&= \frac{3}{2}+\cos 1+\sin 1-\cos 2-2\sin 2
\end{aligned}
$$

3. (11.15:6) The volume can be computed as the double integral of the function $f(x,y)=\dfrac{6-x-2y}{3}$ over region $S=\{(x,y)|0\le x\le 6, 0\le y\le (6-x)/2\}$:

$$
\begin{aligned}
\int\int_S \frac{6-x-2y}{3}\,dy\,dx &= \int_0^6\left(\int_0^{\frac{6-x}{2}} \frac{6-x-2y}{3}\,dy\right)dx = \\
\int_0^6 \left[\frac{6-x}{3}y-\frac{y^2}{3}\right]_{y=0}^{\frac{6-x}{2}} dx &= \int_0^6\left[\frac{(6-x)^2}{12}\right]dx = \left[-\frac{(6-x)^3}{36}\right]_0^6 = 6
\end{aligned}
$$

4. (11.15:13) The domain we integrate over is given as

$$
S=\{-6\le x\le 2,\ \frac{x^2-4}{4}\le y\le 2-x\}
$$

Observe the points of intersection of the two functions of x are at $(-6, 8)$ and $(2, 0)$. Integrating in x first will require dividing the domain into two regions, as on $0 \le y \le 8$, $-\sqrt{4+4y} \le x \le 2-y$ while on $-1 \le y \le 0$ we see $-\sqrt{4+4y} \le x \le \sqrt{4+4y}$.

Therefore we can evaluate our integral

$$\int_{-6}^{2}\int_{\frac{x^2-4}{4}}^{2-x} f(x,y)\,dy\,dx = \int_{-1}^{0}\int_{-\sqrt{4y+4}}^{\sqrt{4y+4}} f(x,y)\,dx\,dy + \int_{0}^{8}\int_{-\sqrt{4y+4}}^{2-y} f(x,y)\,dx\,dy$$

5. (11.18:10) Place the coordinate system so that the sides of the rectangle become parallel to the axis and $A = (0,0)$, $B = (0,b)$, $C = (a,b)$ and $D = (a,0)$. The side AB then is along the y axis and the side AD is along the x axis. The rectangle can be described as $Q = \{0 \le x \le a, 0 \le y \le b\}$. The distances of any point (x,y) from segment AB and AD are x and y respectively. Thus, density $f(x,y)$ and mass $m(Q)$ can be defined as:

$$\begin{aligned} f(x,y) &= x \times y \\ m(Q) &= \int\int_Q f(x,y)\,dy\,dx = \left(\frac{ab}{2}\right)^2 \end{aligned}$$

Then the coordinates of the center of mass can be computed as:

$$\begin{aligned} \overline{x} &= \frac{1}{m(Q)}\int\int_Q x(xy)\,dy\,dx = \frac{2}{3}a \\ \overline{y} &= \frac{1}{m(Q)}\int\int_Q y(xy)\,dy\,dx = \frac{2}{3}b \end{aligned}$$

6. Let f_S, f_R represent the density functions for S, R respectively. We define

$$f_{R\cup S}(\mathbf{x}) = \begin{cases} f_R(\mathbf{x}) & \text{if } \mathbf{x} \in R \\ f_S(\mathbf{x}) & \text{if } \mathbf{x} \in S \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Then

$$\overline{x}_T = \frac{\int\int_{R\cup S} x f_{R\cup S}\,dx\,dy}{\int\int_{R\cup S} f_{R\cup S}\,dx\,dy} = \frac{\int\int_R x f_R\,dx\,dy + \int\int_S x f_S\,dx\,dy}{\int\int_R f_R\,dx\,dy + \int\int_S f_S\,dx\,dy}.$$

Now observe that $\int\int_R x f_R \, dx \, dy = \overline{x}_R \int\int_R f_R \, dx \, dy = x_R m(R)$ and $\int\int_S x f_S \, dx \, dy = \overline{x}_S \int\int_S f_S \, dx \, dy = x_S m(S)$. Thus

$$\overline{x}_T = \frac{\overline{x}_R m(R) + \overline{x}_S m(S)}{m(R) + m(S)}.$$

A similar argument works for $\overline{y}_T$ and the result follows immediately.

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