MODEL ANSWERS TO HWK #10
(18.022 FALL 2010)

(1) We want to show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. We show that $I = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ by proving that

$$I^2 = \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dxdy = \frac{\pi}{4}.$$

(i) The square of side a defined by $0 < x < a$ and $0 < y < a$ contains Q_a, the quarter circle of radius a, and thus:

$$\int_0^a \int_0^a e^{-(x^2+y^2)} dxdy \geq \iint_{Q_a} e^{-(x^2+y^2)} dxdy = \int_0^{\frac{\pi}{2}} \int_0^a e^{-r^2} r dr d\theta = \frac{\pi}{2} \left(\frac{-e^{-r^2}}{2} \bigg|_{r=0}^{a} \right) = \frac{\pi}{4}(1 - e^{-a^2})$$

Taking the limit as a goes to ∞, we conclude that $I^2 \geq \frac{\pi}{4}$.

(ii) The square of side a defined by $0 < x < a$ and $0 < y < a$ is contained in $Q_{\sqrt{2}a}$, the quarter circle of radius $\sqrt{2}a$, and thus:

$$\int_0^a \int_0^a e^{-(x^2+y^2)} dxdy \leq \iint_{Q_{\sqrt{2}a}} e^{-(x^2+y^2)} dxdy = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}a} e^{-r^2} r dr d\theta = \frac{\pi}{2} \left(\frac{-e^{-r^2}}{2} \bigg|_{r=0}^{\sqrt{2}a} \right) = \frac{\pi}{4}(1 - e^{-2a^2})$$

Taking the limit as a goes to ∞, we conclude that $I^2 \leq \frac{\pi}{4}$.

(2) (5.5.15)

$$\iint_D (x^2+y^2)^{3/2} dA = \int_0^{2\pi} \int_0^3 r^3 r dr d\theta = 2\pi \left(\frac{r^5}{5} \bigg|_{r=0}^{3} \right) = \frac{486}{5}\pi$$

(3) (5.5.16)

$$\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} e^{x^2+y^2} dxdx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^a e^{r^2} r dr d\theta = \pi \left(\frac{e^{r^2}}{2} \bigg|_{r=0}^{a} \right) = \frac{\pi}{2}(e^{a^2} - 1)$$

(4) (5.5.20)

$$\int_{\text{rose}} dA = 4 \int_{\text{leaf}} dA = 4 \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r dr d\theta = 4 \int_0^{\frac{\pi}{2}} \frac{(\sin 2\theta)^2}{2} d\theta$$
$$= 2 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2} d\theta = \frac{\pi}{2} - \frac{1}{4} \left(\sin 4\theta \big|_{\theta=0}^{\frac{\pi}{2}} \right) = \frac{\pi}{2}$$

(5) (5.5.21) The cardioid and the circle intersect at the points $(0, 1)$ and $(0, -1)$, and since we want the area *inside* the cardioid and *outside* the circle, the bounds of integration for θ must

be $\pi/2$ to $3\pi/2$.

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{1}^{1-\cos\theta} r dr d\theta &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}((1-\cos\theta)^2-1^2)d\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\cos^2\theta - 2\cos\theta)d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\frac{1+\cos 2\theta}{2} - 2\cos\theta)d\theta \\ &= \frac{\pi}{4}+\frac{1}{8}(\sin 3\pi - \sin\pi) - (sin\frac{3\pi}{2} - \sin\frac{\pi}{2}) = \frac{\pi}{4}+2\end{aligned}$$

(6) (5.5.22)

$$\int_0^{2\pi}\int_0^{3\theta} r dr d\theta = \int_0^{2\pi}\frac{9\theta^2}{2}d\theta = \frac{9}{2}\left(\frac{\theta^3}{3}\Big|_{\theta=0}^{2\pi}\right. = \frac{9}{2}\,\frac{8\pi^3}{3} = 12\pi^3$$

(7) (5.5.29)

$$\begin{aligned}\iiint_W dV &= \int_0^{2\pi}\int_0^1\int_{-\sqrt{10-2r^2}}^{\sqrt{10-2r^2}} r dz dr d\theta = 2\pi\int_0^1 2\sqrt{10-2r^2}r dr \\ &= -\pi\left(\frac{(10-2r^2)^{3/2}}{3/2}\Big|_{r=0}^{1}\right. = \frac{4\sqrt{2}}{3}\pi(5\sqrt{5}-8)\end{aligned}$$

(8) (5.5.30)

$$\iiint_W dV = \int_0^{2\pi}\int_0^2\int_0^{9-r^2} r dz dr d\theta = 2\pi\int_0^2(9r-r^3)dr = 2\pi\left(\frac{9r^2}{2}-\frac{r^4}{4}\Big|_{r=0}^{2}\right. = 28\pi$$

(9) (5.5.31)

$$\begin{aligned}\iiint_W (2+x^2+y^2)dV &= \int_0^{2\pi}\int_3^5\int_0^{\sqrt{25-z^2}} r dz dr d\theta = 2\pi\int_3^5\left(r^2+\frac{r^4}{4}\Big|_{r=0}^{\sqrt{25-z^2}}\right. dz \\ &= \int_3^5(25-z^2+\frac{625-50z^2+z^4}{4})dz = \int_3^5(\frac{725}{4}-\frac{23}{2}z^2+\frac{z^4}{4})dz \\ &= \frac{656}{5}\pi\end{aligned}$$

(10) (5.5.32) By symmetry reasons, the total volume of the solid is 16 times the volume of the portion defined by $z > 0$, $x > 0$ and $0 < y < x$: this sixteenth of the solid is bounded on the bottom by the plane $z = 0$, on the sides by the planes $y = 0$ and $y = x$ and by the cylinder $x^2 + y^2 = a^2$, and on top by $x^2 + z^2 = a^2$. We write the integral in cylindrical coordinates:

$$
\begin{aligned}
\text{Volume} &= 16\int_0^{\frac{\pi}{4}}\int_0^a\int_0^{\sqrt{a^2-r^2\cos^2\theta}} r\,dz\,dr\,d\theta \\
&= 16\int_0^{\frac{\pi}{4}}\int_0^a \sqrt{a^2-r^2\cos^2\theta}\, r\,dr\,d\theta \\
&= \frac{16}{3}\int_0^{\frac{\pi}{4}} \left(-\frac{(a^2-r^2\cos^2\theta)^{\frac{3}{2}}}{\cos^2\theta}\right|_{r=0}^{a} d\theta \\
&= \frac{16}{3}\int_0^{\frac{\pi}{4}} \frac{1-\sin^3\theta}{\cos^2\theta}d\theta \\
&= \frac{16}{3}\left(\int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta}d\theta - \int_0^{\frac{\pi}{4}} \frac{\sin^3\theta}{\cos^2\theta}d\theta\right) \\
&= \frac{16}{3}\left((\tan\theta|_{\theta=0}^{\frac{\pi}{4}} + \int_{\frac{\sqrt{2}}{2}}^{1} \frac{t^2-1}{t^2}dt\right) \\
&= \frac{16}{3}\left(1 + \int_{\frac{\sqrt{2}}{2}}^{1} 1-\frac{1}{t^2}dt\right) \\
&= \frac{16}{3}\left(1 + (t|_{t=\frac{\sqrt{2}}{2}}^{1} + \left(\frac{1}{t}\right|_{t=\frac{\sqrt{2}}{2}}^{1}\right) \\
&= \frac{16}{3}\left(1+1-\frac{\sqrt{2}}{2}+1-\sqrt{2}\right) \\
&= (16-8\sqrt{2})a^3
\end{aligned}
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