18.02 Practice Final A Solutions

1 $P:(1,1,-1)$ $\quad \vec{PQ}=\langle 0,1,1\rangle$
$Q:(1,2,0)$ $\quad \vec{PR}=\langle -3,1,3\rangle$
$R:(-2,2,2)$

$$\vec{PQ}\times\vec{PR}=\begin{vmatrix} i & j & k\\ 0 & 1 & 1\\ -3 & 1 & 3\end{vmatrix}=\langle 2,-3,3\rangle$$

Plane: $\boxed{2x-3y+3z=-4}$

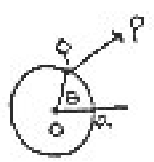
(substitute any of the pts. into $2x-3y+3z=d$)

2 $\begin{vmatrix} 1 & 0 & c\\ 2 & c & 1\\ 1 & -1 & 2\end{vmatrix}=(2c-2c)-(c^2-1)$
$=1-c^2$

$\therefore |\ |=0 \iff \boxed{c=\pm 1}$

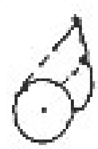
$\begin{bmatrix} 1 & 0 & 2\\ 2 & 2 & 1\\ 1 & -1 & 2\end{bmatrix}$ cofactor $=-\begin{vmatrix} 1 & 0\\ 1 & -1\end{vmatrix}=1$
det $=1-2^2=-3$
$\therefore \boxed{-1/3}$

3



$\vec{OP}=\vec{OQ}+\vec{QP}$
$\vec{OQ}=a\langle\cos\theta,\sin\theta\rangle$
$\vec{QP}=a\theta\left\langle\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right\rangle$

$\therefore x=a\left(\cos\theta+\frac{\theta\sqrt{2}}{2}\right)$
$y=a\left(\sin\theta+\frac{\theta\sqrt{2}}{2}\right)$



4 $\vec{r}=\langle 3\cos t, 5\sin t, 4\cos t\rangle$
$\vec{v}=\langle -3\sin t, 5\cos t, -4\sin t\rangle$
$|\vec{v}|=\sqrt{9\sin^2 t+16\sin^2 t+25\cos^2 t}$
$=\boxed{5}$

Passes through yz plane when $x=0$,
$\therefore$ when $\cos t=0$: $t=\pi/2, 3\pi/2$
$\therefore$ at $\langle 0,\pm 5,0\rangle$

5 $w=x^2y-xy^3$, $\quad P=(2,1)$

a) $\vec{\nabla w}=(2xy-y^3)\hat{\imath}+(x^2-3xy^2)\hat{\jmath}$
$(\vec{\nabla w})_P=3\hat{\imath}-2\hat{\jmath}$
$\left(\frac{dw}{ds}\right)_P=(3\hat{\imath}-2\hat{\jmath})\cdot\frac{(3\hat{\imath}+4\hat{\jmath})}{5}=\boxed{\frac{1}{5}}$

b) $\frac{\Delta w}{\Delta s}\approx\frac{1}{5}$, $\quad\therefore \Delta w\approx\frac{1}{5}(.01)=\boxed{.002}$

6 $x^2+2y^2+2z^2=5$
$\vec{\nabla w}=\langle 2x,4y,4z\rangle=\langle 2,4,4\rangle$ at $(1,1,1)$
tan plane: $\boxed{x+2y+2z=5}$

dihedral angle θ: $\cos\theta=\frac{\langle 1,2,2\rangle\cdot\hat{k}}{3}=\frac{2}{3}$
($\measuredangle$ between normals)

$\boxed{\theta=\cos^{-1}(2/3)}$

7 Minimize $x^2+y^2+z^2$, with $2x+y-z-6=0$ $\circledast$
Lagrange equations:
$2x=2\lambda$ $\quad$ substituting into $\circledast$:
$2y=\lambda$ $\quad 2\lambda+\frac{\lambda}{2}-\left(\frac{-\lambda}{2}\right)=6$
$2z=-\lambda$ $\quad\therefore \lambda=2$

Ans: $(2,1,-1)$

8 $g(x,y,z)=3$ $\quad (\nabla g)_P=\langle 2,-1,-1\rangle$
$\therefore g_x+g_z\cdot\frac{\partial z}{\partial x}=0$; at P, $\frac{\partial z}{\partial x}=-\frac{g_x}{g_z}=\frac{-2}{-1}$
$=\boxed{2}$ Ans.

$$\left(\frac{\partial w}{\partial x}\right)_y=\underset{1}{(f_x)}\underset{1}{\left(\frac{\partial x}{\partial x}\right)_y}+\underset{1}{(f_y)}\underset{0}{\left(\frac{\partial y}{\partial x}\right)_y}+\underset{2}{(f_z)}\underset{2}{\left(\frac{\partial z}{\partial x}\right)_y}$$
$=\boxed{5}$

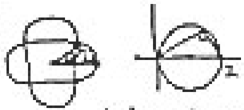
9

$$\int_0^3\int_{x^2}^9 xe^{-y^2}\,dy\,dx$$
$$=\int_0^9\int_0^{\sqrt{y}} xe^{-y^2}\,dx\,dy$$

Inner: $\frac{1}{2}x^2e^{-y^2}\Big]_0^{\sqrt{y}}=\frac{1}{2}ye^{-y^2}$

Outer: $-\frac{e^{-y^2}}{4}\Big]_0^9=\frac{1}{4}\left[1-e^{-81}\right]$

$\boxed{10}$



Circle is $r = 2\cos\theta$

Integrate over $1/8$ region:

$$8\int_0^{\pi/4}\int_0^{2\cos\theta} r^2\cdot r\,dr\,d\theta \qquad \left[\text{or: } 4\int_{-\pi/4}^{\pi/4}\int\cdots\right]$$

$\boxed{11}$ $\oint P\,dy - Q\,dx$ $\quad$ [or: $\oint -Q\,dx + P\,dy$]

b) By Green's Thm: above

$$= \iint_R (P_x + Q_y)\,dx\,dy = \iint_R (a+b)\,dx\,dy$$

$=$ area of R $\Leftrightarrow$ $\boxed{a+b=1}$

$\boxed{12}$

$$F = G\iiint \frac{\cos\varphi}{\rho^2}\cdot\delta\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$\delta = z = \rho\cos\varphi$

$$\therefore F = G\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cos^2\varphi\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= G\cdot 2\pi\cdot\int_0^{\pi/2}\cos^2\varphi\sin\varphi\,d\varphi\cdot\int_0^1\rho\,d\rho$$

$$= G\cdot 2\pi\cdot\left.\frac{-\cos^3\varphi}{3}\right]_0^{\pi/2}\cdot\left.\frac{1}{2}\rho^2\right]_0^1$$

$$= 2\pi G\cdot\frac{1}{3}\cdot\frac{1}{2} = \boxed{\frac{\pi G}{3}}$$

$\boxed{13}$ Line from $P:(1,1,1)$ to $Q:(2,4,8)$ is: $x = 1+t$, $y = 1+3t$, $z = 1+7t$ (since $\vec{PQ} = \langle 1,3,7\rangle$). $\quad 0\le t\le 1$

$$\therefore \int_C (y-x)\,dx + (y-z)\,dz = \int_0^1 2t\,dt - 4\cdot 7t\,dt$$

$$= \int_0^1 -26t\,dt = \left.-13t^2\right]_0^1 = \boxed{-13}$$

$\boxed{14}$ a) $\vec{F} = \langle ay^2,\ 2yx+2yz,\ by^2+z^2\rangle$

Test: $2ay = 2y$ $\therefore a=1$
$2y = 2by$ $\therefore b = 1$
$0 = 0$ ✓

b) By any method, $f(x,y,z) = \boxed{xy^2 + y^2 z + \frac{z^3}{3}}$

c) Any surface S: (contour surface of f) $\boxed{xy^2 + y^2 z + \frac{z^3}{3} = C}$

$\boxed{15}$

$$\oiint_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}\vec{F}\,dV$$

$$\therefore \iint_B \vec{F}\cdot d\vec{S} + \iint_U \vec{F}\cdot d\vec{S} = \iiint_V 3\,dV = 3\cdot\text{vol } V.$$

Volume $V = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \pi/2$

$\iint_B = 0$ since $\vec{F}\cdot\hat{n} = z = 0$ on xy-plane

$$\therefore \iint_U \vec{F}\cdot d\vec{S} = \boxed{3\pi/2}$$

$\boxed{16}$ $\vec{F} = \langle x,y,z\rangle \qquad z = 1-x^2-y^2$

$\hat{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy = \langle 2x, 2y, 1\rangle\,dx\,dy$

$\vec{F}\cdot\hat{n}\,dS = (2x^2+2y^2+z)\,dx\,dy$
$= (x^2+y^2+1)\,dx\,dy$

$\therefore$ flux over U is:

$$\iint_R (x^2+y^2+1)\,dx\,dy = \int_0^{2\pi}\int_0^1 (r^2+1)\,r\,dr\,d\theta$$

$$= 2\pi\left[\frac{r^4}{4} + \frac{r^2}{2}\right]_0^1 = 2\pi\cdot\frac{3}{4} = \boxed{\frac{3\pi}{2}}$$

$\boxed{17}$ $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} i & j & k\\ \partial_x & \partial_y & \partial_z\\ x^2 & y^2 & xz\end{vmatrix} = -z\hat{j}$$

The normal vector to $f(x,z) = 0$ is

$$\hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{f_x\hat{i} + f_z\hat{k}}{|\nabla f|}$$

$\therefore \vec{\nabla}\times\vec{F}\cdot\hat{n} = 0$, so $\oint_C \vec{F}\cdot d\vec{r} = 0$

$\boxed{18}$ $\int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}\,dy\,dx$

a) $= \int_0^\infty e^{-x^2}\,dx\int_0^\infty e^{-y^2}\,dy = I\cdot I$

b) $= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta$

$$= \pi/2\cdot\left.\frac{e^{-r^2}}{-2}\right]_0^\infty = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$$

$I^2 = \pi/4$, $\quad\therefore I = \boxed{\sqrt{\pi}/2}$