18.02 Multivariable Calculus
Fall 2007

18.02 Problem Set 8

Due Thursday 11/1/07, 12:45 pm.

Part A (12 points)

Lecture 21. Thu Oct. 25 Vector fields and line integrals in the plane.
  Read: Notes V1; 15.2 pp. 1023–1028
  Work: 4A/ 1b<u>d</u>, 2a<u>bc</u>, 3a<u>bc</u>d, <u>4</u>; 4B/ 1<u>ab</u>cd, 2a<u>b</u>.

Lecture 22. Fri Oct. 26 Path independence and conservative fields.
  Read: 15.3 to p. 1033
  Work: 4C/ <u>1</u>, 2, <u>3</u>.

Lecture 23. Tue Oct. 30 Gradient fields and potential functions.
  Read: Notes V2 (and refer to 15.3).
  Work: 4C/ <u>5a</u> (by method 1 of V2), <u>5b</u> (by method 2), <u>6ab</u> (by both methods).

Part B (15 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Write the names of all the people you consulted or with whom you collaborated and the resources you used.

Problem 1. (Thursday, 3 points)

Consider the vector field $\vec{F} = (x^2 y + \frac{1}{3}y^3)\,\hat{\imath}$, and let C be the portion of the graph $y = f(x)$ running from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ (assume that $x_1 < x_2$, and f takes positive values). Show that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is equal to the polar moment of inertia of the region R lying below C and above the x-axis (with density $\delta = 1$).

Problem 2. (Friday, 8 points: 1+2+2+1+2)

Consider the vector field $\vec{F}(x, y) = \dfrac{-y\,\hat{\imath} + x\,\hat{\jmath}}{x^2 + y^2}$.

a) Show that $\vec{F}$ is the gradient of the polar angle function $\theta(x, y) = \tan^{-1}(y/x)$ defined over the right half-plane $x > 0$. (Note: this formula for θ does not make sense for $x = 0$!)

b) Suppose that C is a smooth curve in the right half-plane $x > 0$ joining two points $A : (x_1, y_1)$ and $B : (x_2, y_2)$. Express $\int_C \vec{F} \cdot d\vec{r}$ in terms of the polar coordinates (r_1, θ_1) and (r_2, θ_2) of A and B.

c) Compute directly from the definition the line integrals $\int_{C_1} \vec{F} \cdot d\vec{r}$ and $\int_{C_2} \vec{F} \cdot d\vec{r}$, where C_1 is the upper half of the unit circle running from $(1, 0)$ to $(-1, 0)$, and C_2 is the lower half of the unit circle, also going from $(1, 0)$ to $(-1, 0)$.

d) (Tuesday) Show that the curl of $\vec{F}$ is zero at any point of the plane where $\vec{F}$ is defined (not just in the right half-plane $x > 0$).

e) (Tuesday) Using the results of parts (a)-(d), is $\vec{F}$ conservative (path-independent) over its entire domain of definition? Is it conservative over the right half-plane $x > 0$? Justify your answers.

Note: in fact it is true that $\vec{F} = \nabla\theta$ everywhere. However, the polar angle θ cannot be defined as a single-valued differentiable function everywhere (if you try, you will find that it is only well-defined up to adding multiples of 2π). This is why in parts (a) and (b) we only consider the right half-plane; any other region over which θ can be defined unambiguously in a continuous manner would be equally suitable.

Problem 3. (Tuesday, 4 points: 2+2)

a) Calculate the curl of $\vec{F} = r^n(x\,\hat{\imath} + y\,\hat{\jmath})$ (where $r = \sqrt{x^2 + y^2}$; start by finding formulas for r_x and r_y).

b) Whenever possible, find a potential g such that $\vec{F} = \nabla g$. (Hint: look for a potential of the form $g = g(r)$, with $r = \sqrt{x^2 + y^2}$. Watch out for a certain negative value of n for which the general formula doesn't work.)