Hi. Welcome back to recitation. In class, Professor Jerison and Professor Miller have taught you a little bit about Taylor series and some of the manipulations you can do with them, and have computed a bunch of examples for you.

So I have three more examples here of functions whose Taylor series are nice to compute. So the first one is cosh x. That's the hyperbolic cosine. So just to remind you, this can be written in terms of the exponential function as e to the x plus e to the minus x over 2.

The second one is the function 2 times sine of x times cosine of x, just your regular sine and cosine here.

And the third one is x times the logarithm of the quantity 1 minus x cubed.

So why don't you pause the video, take some time to work out the Taylor series for these three functions, come back, and we can work them out together.

So here we have three functions whose Taylor series we're trying to compute. Let's start with the first one and go from there.

So this first one is the hyperbolic cosine that's given by the formula e to the x plus e to the minus x over 2. So there are a couple different ways you could go about this one. This is actually, the hyperbolic cosine is very susceptible to the method of just using the formula that you have. So if you remember, the derivative of the hyperbolic cosine is the hyperbolic sine. The derivative of the hyperbolic sine is the hyperbolic cosine again. So this function has very easy-to-understand derivatives, which you can see, you know, just by looking at its formula. It's easy to understand, because the exponential function has very simple derivatives, and e to the the minus x also has very simple derivatives.

So you could do it like that. The other thing you could do, is that you already know the Taylor series for e to the x. And I believe you've also seen the Taylor series for e to the minus x, and even if you haven't, you can figure it out just by substitution. So if you remember, so e to the x is given by the sum from n equals 0 to infinity of x to the n over n factorial. I'm going to pull the 1/2 out in front. And e to the minus x is given by the same thing, if you put in minus x for x. So it's n equals 0 to infinity, so that works out to minus 1 to the n x to the n or n factorial.

Now, when you add these two series together, what you see is that when n is even, over here, you have x to the n over n factorial, and over here, you have x to the n over n factorial. So what you get is, well, you get 2 x to the n over n factorial, and then you multiply by a half, so you just get x to the n over n factorial.

When n is odd, here you have x to the n over n factorial, and here you have minus 1 x to the n over n factorial. So you add them and you get 0.

So what happens is that this series looks just like the series for e to the x, except the odd terms have died off. So we're left with just 1 plus x squared over 2 factorial plus x to the fourth over 4 factorial plus x to the sixth over 6 factorial, and so on. And if you wanted to write this in summation notation, you could write it as the sum from n equals 0 to infinity of x to the 2n over 2n factorial.

So this is the Taylor series for the hyperbolic cosine function. Also, if you wanted, say, the hyperbolic sine function, you could do something very similar, or you could remember that the hyperbolic sine is the derivative of the hyperbolic cosine, and just take a derivative right from this expression.

One other thing that you should notice is that this looks very similar to the expression of the Taylor series for cosine of x. So more of our sort of funny coincidences between regular trig functions and hyperbolic trig functions.

All right. That's the first one. How about the second one? So here we have just some regular trig functions. We have 2 sine x cosine x. Let me see where I've got some space. I can do it right here. Let me box off a little space for myself. So 2 sine x cosine x-- there are a couple different ways you could proceed with this function. So one is, you know the Taylor series for sine x and cosine x already. So if all you wanted was a few terms of this Taylor series, one natural thing to do would be to take the series for sine x, take the series for cosine x, multiply them together like you would multiply polynomials, and what you would get is the Taylor series for this expression, for this function.

That's one way to proceed. That works perfectly well. Another thing you could do, is you could try taking derivatives. You could have a situation where every time you take a derivative, you apply product rule. It's going to get more and more complicated. It still works. It's a little complicated to do it that way, if you wanted more than just a few terms.

The other thing you could do, is you could remember your trig identities. So if you look at this expression, this should be familiar to you, because it's just sine of 2x. So once you realize that this is sine of 2x, there's a much, much shorter path available to you, which is that you already know the Taylor series for sine of x, so what you can do, is you can just plug in 2x into that Taylor series. So sine of x is-- well, so OK. So sine of x is x-- so in this case, that's going to be 2x-- then minus, so in sine of x, we have x cubed over 3 factorial. So here we're going to have 2x cubed over 3 factorial plus-- OK. So then, you know, and so on. So here we'll have 2x to the fifth over 5 factorial minus-- so on. If you wanted to write this in summation notation, you could write it as the sum from n equals 0 to infinity. Well, the denominator has got to be 2n plus 1 factorial, because we want it to go through the odds. And then we've got minus 1 to the n times 2 to the 2n plus 1 times x to the 2n plus 1. So this is 2x. What we've got here, if you didn't have the 2's there, that would just be the series for the regular sine.

OK. So this is the series for this function, 2 sine x cosine x. And I'll go over here to do the third one. So what is the third one? It's x ln 1 minus x cubed. Well, what can we do with this series? The x out front is just multiplying this logarithm part. That's something we can save until the end. If we can figure out what the Taylor series for the ln of 1 minus x cubed part is, then we just multiply x into it, and that'll give us the Taylor series for this whole thing. So the x out front is pretty simple. So now what about this ln of 1 minus x cubed stuff? Well, a thing to remember is, does it remind you of anything we've done before? Well, we have a Taylor series for a logarithm function, right? We've already seen in lecture, I believe, we've seen that ln of 1 plus x is equal to x minus x squared over 2 plus x cubed over 3, minus x to the fourth over four, and so on, alternating signs. Notice that the denominators, when you have a logarithm, these are not factorials. These are just the integer 2, the integer 3, the integer 4, unlike for exponentials and trig functions.

So this is what log of 1 plus x-- this is the Taylor series for log of 1 plus x. Well, how does that help us? Well, log of 1 minus x cubed we can get from log of 1 plus x, with the appropriate substitution. So in particular, we just have to put minus x cubed in for x here. So what does that give us? It gives us the ln of 1 minus x cubed is equal to-- well, minus x cubed minus, so we put minus x cubed in here, we square it, and we just get x to the sixth. x to the sixth over 2. Then, all right. So minus x cubed quantity cubed is minus x to the ninth. So minus x to the ninth over 3, minus x to the twelfth over 4, and so on.

And so finally, x ln of 1 minus x cubed, we just get by multiplying this whole expression through by x. So this is equal to minus x to the fourth minus x to the seventh over 2 minus x-- whoops, not ten-- minus x to the tenth over 3, minus x to the 13 over 4, and so on. And I'll leave it as an exercise for you to figure out how to write this in summation notation, if you wanted.

So just quickly to summarize, we had these three power series, these three functions that we started out with, and we used a bunch of different techniques that we've learned in order to compute their power series. So over here, we took the function that we'd seen, and we knew a formula for it in terms of other functions that we already knew, and so we plugged in those power series, and used our addition rule for power series. We could have also done this one directly from the definition, if we had wanted to.

For the second one, for the 2 sine x cosine x, we recognized that as something that is susceptible to a substitution, although also, with a little more, work, we could have done it by a couple of different methods. For example, by multiplying two power series together. And finally, for this third one, for the x ln of 1 minus x cubed, we first saw the substitution here that we could make, and then we just did a multiplication by a polynomial, which is a relatively easy thing to do for power series.

So that's what we did in this recitation, and I'll leave it at that.