Indefinite Integrals over Singularities

When computing $\int_0^\infty \frac{dx}{\sqrt{x^2+10}}$ we had to take an extra step to avoid the integral $\int_0^1 \frac{dx}{x}$. We'll now go back and discuss integration near singular points.

Integrals like $\int_0^1 \frac{dx}{x}$ are known as *indefinite integrals of the second type.* Examples include:

$$\int_0^1 \frac{dx}{\sqrt{x}}, \quad \int_0^1 \frac{dx}{x}, \quad \text{and} \quad \int_0^1 \frac{dx}{x^2}.$$

These integrals turn out to be fairly straightforward to calculate:

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{x}} &= \int_0^1 x^{-1/2} dx \\ &= \frac{1}{1/2} x^{1/2} \Big|_0^1 \\ &= 2x^{1/2} \Big|_0^1 \\ &= 2 \cdot 1^{1/2} - 2 \cdot 0^{1/2} \\ &= 2.\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \ln x|_0^1 \\ &= \ln 1 - \ln 0 \quad \text{(diverges.)}\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{dx}{x^2} &= -x^{-1}\Big|_0^1 \\ &= -\frac{1}{1} - \left(-\frac{1}{0}\right) \quad \text{(diverges.)}\end{aligned}$$

However, you can get into trouble if you're not careful. Consider the following calculation:

$$\begin{aligned}\int_{-1}^1 \frac{dx}{x^2} &= -x^{-1}\Big|_{-1}^1 \\ &= -(1^{-1}) - (-(-1)^{-1}) \\ &= -1 - 1 \\ &= -2.\end{aligned}$$

This is ridiculous! As we see from Figure 1, $\frac{1}{x^2}$ is always positive. The area under the graph of $y = \frac{1}{x^2}$ between -1 and 1 is clearly greater than 2; in particular it cannot be a negative number.

In fact, the area under the graph of $y = \frac{1}{x^2}$ between -1 and 1 is infinite, not -2. The calculation above is nonsense.

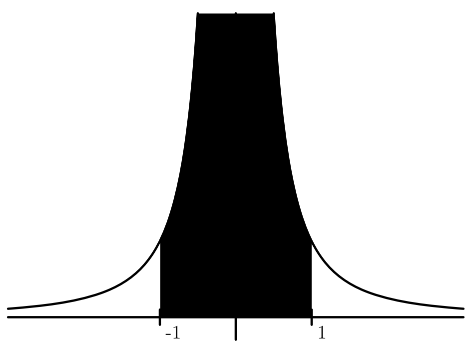

Figure 1: Graph of $y = \frac{1}{x^2}$.