l'Hôpital's Rule, Continued

In keeping with the spirit of "dealing with infinity" we look at an application of l'Hôpital's rule to a limit of the form $\frac{\infty}{\infty}$. In other words, as x approaches a we have:

- $f(x) \to \infty$
- $g(x) \to \infty$
- $\frac{f'(x)}{g'(x)} \to L$

and so we can conclude that:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

(Recall that a and L may be infinite.)

Rates of Growth

We apply this to "rates of growth"; the study of how rapidly functions increase. We know that the functions $\ln x$ and x^2 both go to infinity as x goes to infinity, and that x^2 increases much more rapidly than $\ln x$. We can formalize this idea as follows:

If $f(x) > 0$ and $g(x) > 0$ as x approaches infinity, then

$$f(x) << g(x) \text{ as } x \to \infty \quad \text{means} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

(Read $f(x) << g(x)$ as "$f(x)$ is a lot less than $g(x)$".) In our example, $f(x) = \ln x$ and $g(x) = x^2$. If we use l'Hôpital's rule to evaluate $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\ln x}{x^2}$ we get:

$$\begin{aligned} \lim_{x \to \infty} \frac{\ln x}{x^2} &= \lim_{x \to \infty} \frac{\frac{1}{x}}{2x} \\ &= \lim_{x \to \infty} \frac{1}{2x^2} \\ &= 0. \end{aligned}$$

We conclude that $\ln x << x^2$ as $x \to \infty$.

If $p > 0$ then:

$$\ln x << x^p << e^x << e^{x^2} \text{ as } x \to \infty.$$

Rates of Decay

"Rates of decay" are rates at which functions tend to 0 as x goes to infinity. Again our new notation comes in handy; if $p > 0$ then:

$$\frac{1}{\ln x} >> \frac{1}{x^p} >> e^{-x} >> e^{-x^2} \text{ as } x \to \infty.$$