Path of a Falling Object

A teenager throws a ball off a rooftop. Assume that the x coordinate of the ball is given by $x(t) = t$ meters and its y coordinate satisfies the following properties:

$$\begin{aligned} y''(t) &= -9.8 \text{ meters/second} \\ y'(0) &= 0 \\ y(0) &= 5 \text{ meters.} \end{aligned}$$

a) Find an equation directly describing y in terms of t.

b) Find a parametrization $(x(t), y(t))$ which describes the path of the ball.

c) Find the speed $\frac{ds}{dt}$ of the ball (this answer will only be valid for times before the ball hits the ground.)

Solution

You may wish to begin by drawing a sketch of the situation. The teenager stands at the vertex of what turns out to be the parabolic path of the ball. The ball moves forward at a constant speed of 1 meter per second, and its horizontal position decreases slowly at the beginning and more rapidly at the end.

a) Find an equation directly describing y in terms of t.

Here we've been given the second derivative of a function and some initial conditions and asked to find the equation of the function. Broadly, we find an antiderivative $F(t) + c$ and then use the initial conditions to solve for c. In detail, this looks like:

$$\begin{aligned} y''(t) &= 9.8 \\ y'(t) &= 9.8t + c \\ y'(0) &= 9.8 \cdot 0 + c \\ y'(0) = 0 &\Rightarrow c = 0 \\ y'(t) &= 9.8t \end{aligned}$$

$$\begin{aligned} y'(t) &= 9.8t \\ y(t) &= \frac{1}{2} 9.8t^2 + C \\ y(0) &= 4.9t^2 + C \\ y(0) = 5 &\Rightarrow C = 5 \\ y(t) &= 4.9t^2 + 5. \end{aligned}$$

b) Find a parametrization $(x(t), y(t))$ which describes the path of the ball.

Once we've answered the previous problem, this one is easy. Splitting a problem like this into x and y parts can make it more manageable.

We were given that $x(t) = t$, and we found that $y(t) = 4.9t^2 + 5$, so:

$$(x(t), y(t)) = (t, 4.9t^2 + 5).$$

We might graph this curve to see if our answer is reasonable.

c) Find the speed $\frac{ds}{dt}$ of the ball (this answer will only be valid for times before the ball hits the ground.)

In general,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Here, $\frac{dx}{dt} = x'(t) = 1$ and $\frac{dy}{dt} = y' = 9.8t$. Hence,

$$\begin{aligned} \frac{ds}{dt} &= \sqrt{1^2 + (9.8t)^2} \\ &\approx \sqrt{1 + 96t^2}. \end{aligned}$$

The speed of the ball is 1 meter per second initially, and it accelerates as t increases. After about one second, the ball is moving at a speed of 10 meters per second. This seems plausible. (For comparison, a pitcher might throw a baseball at 40 meters per second.)

For more practice with parametric equations, compute the time and location at which the ball hits the ground ($y(t) = 0$) and the speed at which it is moving at that time.