Introduction to Arc Length

Now that we're done with techniques of integration, we'll return to doing some geometry; this will lead to some of the tools you'll need in multivariable calculus. Our first topic is *arc length*, which is calculated using another cumulative sum which will have an associated story and picture.

Suppose you have a roadway with mileage markers $s_0, s_1, s_2, ..., s_n$ along the road. The distance traveled along the road — the *arc length* — is described by this parameter s. If we look at the road as a graph, we can let a be the x coordinate of the first point s_0 on the curve or road and b be the x coordinate of the end point s_n of the curve, and x_i as the x-coordinate of s_i. This is reminiscent of what we did with Riemann sums.

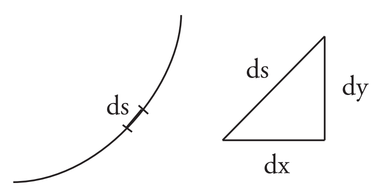

Figure 1: Straight line approximation of arc length.

We'll approximate the length s of the curve by summing the straight line distances between the points s_i. As n increases and the distance between the s_i decreases, the straight line distance from s_i to s_{i-1} will get closer and closer to the distance Δs along the curve. We can use the Pythagorean theorem to see that that distance equals $\sqrt{(\Delta x)^2 + (\Delta y)^2}$. In other words:

$$(\Delta s)^2 \approx \overbrace{(\Delta x)^2 + (\Delta y)^2}^{(\text{hypotenuse})^2}.$$

We apply the tools of calculus to this estimate; in the infinitesimal this is exactly correct:

$$(ds)^2 = (dx)^2 + (dy)^2.$$

In the future we'll omit the parentheses and write this as $ds^2 = dx^2 + dy^2$. These are squares of differentials; try not to mistake them for differentials of squares.

The next thing we do is take the square root:

$$ds = \sqrt{dx^2 + dy^2}.$$

This is the formula that Professor Jerison has memorized, but you can rewrite it in several useful ways. For instance, you can factor out the dx to get:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

This is the form we'll be using today; when we add up all the infinitesimal values of ds we'll find that:

$$\begin{aligned}
\text{Arc Length} &= \text{distance along the curve from } s_0 \text{ to } s_n \\
&= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int ds \\
&= \int_a^b \sqrt{1 + f'(x)^2}\,dx \quad (y = f(x))
\end{aligned}$$

Question: Is $f'(x)^2$ equal to $f''(x)$?

Answer: No. Suppose $f(x) = x^2$. Then $f'(x) = 2x$, $f'(x)^2 = 4x^2$ and $f''(x) = 2$.

Question: What are the limits of integration on $\int ds$, above?

Answer: If you're integrating with respect to s you'll start at s_0 and end at s_n. If you're integrating with respect to a different variable you'll have different limits of integration, as happens when we change variables. The values s_0 and s_n are mileage markers along the road; they're not the same as a and b. When we measure arc length, remember that we're measuring distance along a curved path.